Domains and automata in duality-theoretic form

Sam van Gool

IRIF, Université Paris Cité

International Symposium of Domain Theory and its Applications
Qufu, Shandong, China
2-5 August 2024

Introduction

- First of all, I would like to thank the organizers of this conference for inviting me to speak here, especially Prof. Achim Jung and Prof. Longchun Wang.

Introduction

- First of all, I would like to thank the organizers of this conference for inviting me to speak here, especially Prof. Achim Jung and Prof. Longchun Wang.
- In this talk, I will tell you about uses of **duality** in

Introduction

- First of all, I would like to thank the organizers of this conference for inviting me to speak here, especially Prof. Achim Jung and Prof. Longchun Wang.

- In this talk, I will tell you about uses of **duality** in
  - domain theory, and

Introduction

- First of all, I would like to thank the organizers of this conference for inviting me to speak here, especially Prof. Achim Jung and Prof. Longchun Wang.

- In this talk, I will tell you about uses of **duality** in
  - domain theory, and
  - automata theory.

Introduction

- First of all, I would like to thank the organizers of this conference for inviting me to speak here, especially Prof. Achim Jung and Prof. Longchun Wang.
- In this talk, I will tell you about uses of **duality** in
  - domain theory, and
  - automata theory.
- We will discover a **common property** that pops up in both:

Introduction

- First of all, I would like to thank the organizers of this conference for inviting me to speak here, especially Prof. Achim Jung and Prof. Longchun Wang.

- In this talk, I will tell you about uses of **duality** in
  - domain theory, and
  - automata theory.

- We will discover a **common property** that pops up in both:
  - preserving joins at primes.

Reference

- The contents of this talk are based on our **book**:

Reference

- The contents of this talk are based on our **book**:

  Topological Duality for Distributive Lattices: Theory and Applications,
  by Mai Gehrke and Sam van Gool.
  Cambridge University Press, 369pp (2024).

Reference

- The contents of this talk are based on our **book**:

  Topological Duality for Distributive Lattices: Theory and Applications,
  by Mai Gehrke and Sam van Gool.
  Cambridge University Press, 369pp (2024).

- The topic of the book is **Stone-Priestley duality**, with applications to logic and the foundations of computer science.

Reference

- The contents of this talk are based on our **book**:

  Topological Duality for Distributive Lattices: Theory and Applications,
  by Mai Gehrke and Sam van Gool.
  Cambridge University Press, 369pp (2024).

- The topic of the book is **Stone-Priestley duality**, with applications to logic and the foundations of computer science.
- I will give a bit *more practical information* about the book at the end of the talk.

Overview

Duality

Domains

Automata

Stone duality

- A bounded distributive lattice is a partial order $(L, \leq)$ such that every **finite** subset $S \subseteq L$ has a least upper bound $\bigvee S$ ('*join* of S'), and a greatest lower bound $\bigwedge S$ ('*meet* of S'), and for any $a \in L$, $a \wedge \bigvee S = \bigvee_{s \in S}(a \wedge s)$.

Stone duality

- A bounded distributive lattice is a partial order $(L, \leq)$ such that every **finite** subset $S \subseteq L$ has a least upper bound $\bigvee S$ ('*join* of S'), and a greatest lower bound $\bigwedge S$ ('*meet* of S'), and for any $a \in L$, $a \wedge \bigvee S = \bigvee_{s \in S}(a \wedge s)$.

- M. H. Stone (1936) proved that every bounded distributive lattice L is isomorphic to the lattice of **compact-and-open sets** of some topological space X_L.

Stone duality

- A bounded distributive lattice is a partial order $(L, \leq)$ such that every **finite** subset $S \subseteq L$ has a least upper bound $\bigvee S$ ('*join* of S'), and a greatest lower bound $\bigwedge S$ ('*meet* of S'), and for any $a \in L$, $a \wedge \bigvee S = \bigvee_{s \in S}(a \wedge s)$.
- M. H. Stone (1936) proved that every bounded distributive lattice L is isomorphic to the lattice of **compact-and-open sets** of some topological space X_L.
- Moreover, there is a **unique** such space among the spaces that are *stably compact* and have a *base* of compact-open sets; we call such spaces spectral.

Stone duality

- A bounded distributive lattice is a partial order $(L, \leq)$ such that every **finite** subset $S \subseteq L$ has a least upper bound $\bigvee S$ ('*join* of S'), and a greatest lower bound $\bigwedge S$ ('*meet* of S'), and for any $a \in L$, $a \wedge \bigvee S = \bigvee_{s \in S}(a \wedge s)$.

- M. H. Stone (1936) proved that every bounded distributive lattice L is isomorphic to the lattice of **compact-and-open sets** of some topological space X_L.

- Moreover, there is a **unique** such space among the spaces that are *stably compact* and have a *base* of compact-open sets; we call such spaces spectral.

- Also, homomorphisms $L \to L'$ correspond to *certain continuous* $X_{L'} \to X_L$.

Stably compact spaces

- A saturated set in a topological space is any set that is an intersection of open sets. Equivalently, it is an up-set for the *specialization order*.

Stably compact spaces

- A saturated set in a topological space is any set that is an intersection of open sets. Equivalently, it is an up-set for the *specialization order*.
- A stably compact space is a topological space which is:

Stably compact spaces

- A saturated set in a topological space is any set that is an intersection of open sets. Equivalently, it is an up-set for the *specialization order*.
- A stably compact space is a topological space which is:
  - T_0;

Stably compact spaces

- A saturated set in a topological space is any set that is an intersection of open sets. Equivalently, it is an up-set for the *specialization order*.
- A stably compact space is a topological space which is:
  - T_0;
  - compact;

Stably compact spaces

- A saturated set in a topological space is any set that is an intersection of open sets. Equivalently, it is an up-set for the *specialization order*.
- A stably compact space is a topological space which is:
  - T_0;
  - compact;
  - locally compact;

Stably compact spaces

- A saturated set in a topological space is any set that is an intersection of open sets. Equivalently, it is an up-set for the *specialization order*.
- A stably compact space is a topological space which is:
  - T_0;
  - compact;
  - locally compact;
  - coherent: Finite intersection of compact-saturated is compact;

Stably compact spaces

- A saturated set in a topological space is any set that is an intersection of open sets. Equivalently, it is an up-set for the *specialization order*.
- A stably compact space is a topological space which is:
  - T_0;
  - compact;
  - locally compact;
  - coherent: Finite intersection of compact-saturated is compact;
  - well-filtered: For any filtering collection $\mathcal{F}$ of compact-saturated and any open set U, if $\bigcap \mathcal{F} \subseteq U$ then there exists $K \in \mathcal{F}$ such that $K \subseteq U$.

Stably compact spaces

- A saturated set in a topological space is any set that is an intersection of open sets. Equivalently, it is an up-set for the *specialization order*.
- A stably compact space is a topological space which is:
  - T_0;
  - compact;
  - locally compact;
  - coherent: Finite intersection of compact-saturated is compact;
  - well-filtered: For any filtering collection $\mathcal{F}$ of compact-saturated and any open set U, if $\bigcap \mathcal{F} \subseteq U$ then there exists $K \in \mathcal{F}$ such that $K \subseteq U$.

Proposition

If ρ is a stably compact topology, then the complements of the ρ-compact-saturated are ***also*** *a stably compact topology, ρ^∂.*

Stably compact spaces

- A saturated set in a topological space is any set that is an intersection of open sets. Equivalently, it is an up-set for the *specialization order*.
- A stably compact space is a topological space which is:
  - T_0;
  - compact;
  - locally compact;
  - coherent: Finite intersection of compact-saturated is compact;
  - well-filtered: For any filtering collection $\mathcal{F}$ of compact-saturated and any open set U, if $\bigcap \mathcal{F} \subseteq U$ then there exists $K \in \mathcal{F}$ such that $K \subseteq U$.

Proposition

A T_0 locally compact space is well-filtered if, and only if, it is sober.

From Stone to Priestley: patching stably compact spaces

- If Stone's maxim was that one must 'always **topologize**', then the important work of H. A. Priestley (1970) says that one must not forget the **order**.

From Stone to Priestley: patching stably compact spaces

- If Stone's maxim was that one must 'always **topologize**', then the important work of H. A. Priestley (1970) says that one must not forget the **order**.
- A compact ordered space is a tuple $(X, \leq, \tau)$ such that:

From Stone to Priestley: patching stably compact spaces

- If Stone's maxim was that one must 'always **topologize**', then the important work of H. A. Priestley (1970) says that one must not forget the **order**.
- A compact ordered space is a tuple $(X, \leq, \tau)$ such that:
  - The topological space (X, τ) is **compact**;

From Stone to Priestley: patching stably compact spaces

- If Stone's maxim was that one must 'always **topologize**', then the important work of H. A. Priestley (1970) says that one must not forget the **order**.
- A compact ordered space is a tuple $(X, \leq, \tau)$ such that:
  - The topological space (X, τ) is **compact**;
  - The partial order $\leq$ is **closed** in $X \times X$.

From Stone to Priestley: patching stably compact spaces

- If Stone's maxim was that one must 'always **topologize**', then the important work of H. A. Priestley (1970) says that one must not forget the **order**.
- A compact ordered space is a tuple $(X, \leq, \tau)$ such that:
  - The topological space (X, τ) is **compact**;
  - The partial order $\leq$ is **closed** in $X \times X$.
- Given (X, ρ) stably compact, we **obtain** a compact ordered space $(X, \leq, \tau)$, where

From Stone to Priestley: patching stably compact spaces

- If Stone's maxim was that one must 'always **topologize**', then the important work of H. A. Priestley (1970) says that one must not forget the **order**.
- A compact ordered space is a tuple $(X, \leq, \tau)$ such that:
  - The topological space (X, τ) is **compact**;
  - The partial order $\leq$ is **closed** in $X \times X$.
- Given (X, ρ) stably compact, we **obtain** a compact ordered space $(X, \leq, \tau)$, where
  - $\leq$ is the *specialization order* of ρ;

From Stone to Priestley: patching stably compact spaces

- If Stone's maxim was that one must 'always **topologize**', then the important work of H. A. Priestley (1970) says that one must not forget the **order**.
- A compact ordered space is a tuple $(X, \leq, \tau)$ such that:
  - The topological space (X, τ) is **compact**;
  - The partial order $\leq$ is **closed** in $X \times X$.
- Given (X, ρ) stably compact, we **obtain** a compact ordered space $(X, \leq, \tau)$, where
  - $\leq$ is the *specialization order* of ρ;
  - τ is the patch topology $\rho \vee \rho^{\partial}$, i.e., the smallest topology containing both ρ and ρ^{∂}.

From Stone to Priestley: patching stably compact spaces

- If Stone's maxim was that one must 'always **topologize**', then the important work of H. A. Priestley (1970) says that one must not forget the **order**.
- A compact ordered space is a tuple $(X, \leq, \tau)$ such that:
  - The topological space (X, τ) is **compact**;
  - The partial order $\leq$ is **closed** in $X \times X$.
- Given (X, ρ) stably compact, we **obtain** a compact ordered space $(X, \leq, \tau)$, where
  - $\leq$ is the *specialization order* of ρ;
  - τ is the patch topology $\rho \vee \rho^{\partial}$, i.e., the smallest topology containing both ρ and ρ^{∂}.

Proposition

*The topology ρ contains exactly the **τ-open up-sets**.*

From Stone to Priestley: patching stably compact spaces

- If Stone's maxim was that one must 'always **topologize**', then the important work of H. A. Priestley (1970) says that one must not forget the **order**.
- A compact ordered space is a tuple $(X, \leq, \tau)$ such that:
  - The topological space (X, τ) is **compact**;
  - The partial order $\leq$ is **closed** in $X \times X$.
- Given (X, ρ) stably compact, we **obtain** a compact ordered space $(X, \leq, \tau)$, where
  - $\leq$ is the *specialization order* of ρ;
  - τ is the patch topology $\rho \vee \rho^{\partial}$, i.e., the smallest topology containing both ρ and ρ^{∂}.

Proposition

*The assignment $(X, \rho) \mapsto (X, \leq, \tau)$ gives an **isomorphism** of categories of stably compact spaces and compact ordered spaces (* **KOrd** *).*

From Stone to Priestley: patching stably compact spaces

- If Stone's maxim was that one must 'always **topologize**', then the important work of H. A. Priestley (1970) says that one must not forget the **order**.
- A compact ordered space is a tuple $(X, \leq, \tau)$ such that:
  - The topological space (X, τ) is **compact**;
  - The partial order $\leq$ is **closed** in $X \times X$.
- Given (X, ρ) stably compact, we **obtain** a compact ordered space $(X, \leq, \tau)$, where
  - $\leq$ is the *specialization order* of ρ;
  - τ is the patch topology $\rho \vee \rho^{\partial}$, i.e., the smallest topology containing both ρ and ρ^{∂}.

Proposition

*The assignment $(X, \rho) \mapsto (X, \leq, \tau)$ gives an **isomorphism** of categories of spectral and Priestley spaces (**Pr**).*

Morphisms and total order disconnectedness

- I just made claims about *categories* without saying what the morphisms are.
  Let me rectify that now:

Morphisms and total order disconnectedness

- I just made claims about *categories* without saying what the morphisms are.
  Let me rectify that now:
- For *compact ordered spaces*: morphisms are continuous **monotone** functions.

Morphisms and total order disconnectedness

- I just made claims about *categories* without saying what the morphisms are.
  Let me rectify that now:
- For *compact ordered spaces*: morphisms are continuous **monotone** functions.
- For (X, ρ_X) and (Y, ρ_Y) *stably compact*: morphisms are functions $f : X \to Y$ which are coherent, i.e., continuous $(X, \rho_X) \to (Y, \rho_Y)$ **and** $(X, \rho_X^\partial) \to (Y, \rho_Y^\partial)$.

Morphisms and total order disconnectedness

- I just made claims about *categories* without saying what the morphisms are. Let me rectify that now:
- For *compact ordered spaces*: morphisms are continuous **monotone** functions.
- For (X, ρ_X) and (Y, ρ_Y) *stably compact*: morphisms are functions $f : X \to Y$ which are coherent, i.e., continuous $(X, \rho_X) \to (Y, \rho_Y)$ **and** $(X, \rho_X^\partial) \to (Y, \rho_Y^\partial)$.
- A Priestley space is a compact ordered space $(X, \leq, \tau)$ such that: For every $x, y \in X$, if $x \not\leq y$, then there exists a **clopen up-set** K such that $x \in K$ and $y \notin K$ (totally order-disconnected).

Morphisms and total order disconnectedness

- I just made claims about *categories* without saying what the morphisms are. Let me rectify that now:
- For *compact ordered spaces*: morphisms are continuous **monotone** functions.
- For (X, ρ_X) and (Y, ρ_Y) *stably compact*: morphisms are functions $f : X \to Y$ which are coherent, i.e., continuous $(X, \rho_X) \to (Y, \rho_Y)$ **and** $(X, \rho_X^\partial) \to (Y, \rho_Y^\partial)$.
- A Priestley space is a compact ordered space $(X, \leq, \tau)$ such that: For every $x, y \in X$, if $x \not\leq y$, then there exists a **clopen up-set** K such that $x \in K$ and $y \notin K$ (totally order-disconnected).

Proposition

*The assignment $(X, \rho) \mapsto (X, \leq, \tau)$ gives an **isomorphism** between the categories of stably compact spaces and compact ordered spaces (**KOrd**).*
The spectral spaces are exactly those which are sent to Priestley spaces.

Stone-Priestley duality for distributive lattices

Theorem

The category of bounded distributive lattices is ***dually equivalent*** *to the category of Priestley spaces, and therefore also to the category of spectral spaces.*

Stone-Priestley duality for distributive lattices

Theorem

The category of bounded distributive lattices is ***dually equivalent*** *to the category of Priestley spaces, and therefore also to the category of spectral spaces.*

- For any bounded distributive lattice L, define the dual space $(X_L, \leq, \tau)$:

Stone-Priestley duality for distributive lattices

Theorem

The category of bounded distributive lattices is ***dually equivalent*** *to the category of Priestley spaces, and therefore also to the category of spectral spaces.*

- For any bounded distributive lattice L, define the dual space $(X_L, \leq, \tau)$:
  - points of X_L are bounded lattice homomorphisms $L \to 2$;

Stone-Priestley duality for distributive lattices

Theorem

The category of bounded distributive lattices is ***dually equivalent*** *to the category of Priestley spaces, and therefore also to the category of spectral spaces.*

- For any bounded distributive lattice L, define the dual space $(X_L, \leq, \tau)$:
  - points of X_L are bounded lattice homomorphisms $L \to 2$;
  - $\leq$ is the point-wise order;

Stone-Priestley duality for distributive lattices

Theorem

*The category of bounded distributive lattices is **dually equivalent** to the category of Priestley spaces, and therefore also to the category of spectral spaces.*

- For any bounded distributive lattice L, define the dual space $(X_L, \leq, \tau)$:
  - points of X_L are bounded lattice homomorphisms $L \to 2$;
  - $\leq$ is the point-wise order;
  - τ is generated by the family of sets $\widehat{a}$ and $\widehat{a}^c$, as a ranges over L, where:

$$\widehat{a} \stackrel{\text{def}}{=} \{x \in X \mid x(a) = 1\} \text{ and } \widehat{a}^c = \{x \in X \mid x(a) = 0\}$$

Stone-Priestley duality for distributive lattices

Theorem

*The category of bounded distributive lattices is **dually equivalent** to the category of Priestley spaces, and therefore also to the category of spectral spaces.*

- For any bounded distributive lattice L, define the dual space $(X_L, \leq, \tau)$:
  - points of X_L are bounded lattice homomorphisms $L \to 2$;
  - $\leq$ is the point-wise order;
  - τ is generated by the family of sets $\widehat{a}$ and $\widehat{a}^c$, as a ranges over L, where:

$$\widehat{a} \stackrel{\text{def}}{=} \{x \in X \mid x(a) = 1\} \text{ and } \widehat{a}^c = \{x \in X \mid x(a) = 0\}$$

- Then $(X, \leq, \tau)$ is a Priestley space.

Stone-Priestley duality for distributive lattices

Theorem

The category of bounded distributive lattices is ***dually equivalent*** *to the category of Priestley spaces, and therefore also to the category of spectral spaces.*

- For any bounded distributive lattice L, define the dual space $(X_L, \leq, \tau)$:
  - points of X_L are bounded lattice homomorphisms $L \to 2$;
  - $\leq$ is the point-wise order;
  - τ is generated by the family of sets $\widehat{a}$ and $\widehat{a}^c$, as a ranges over L, where:

$$\widehat{a} \stackrel{\mathrm{def}}{=} \{x \in X \mid x(a) = 1\} \text{ and } \widehat{a}^c = \{x \in X \mid x(a) = 0\}$$

- Then $(X, \leq, \tau)$ is a Priestley space.
- For $h\colon L' \to L$, define $f_h\colon X_L \to X_{L'}$ by $f_h(x) \stackrel{\mathrm{def}}{=} x \circ h$.

Stone-Priestley duality for distributive lattices

Theorem

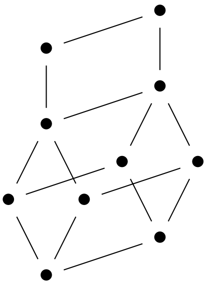
*The category of bounded distributive lattices is **dually equivalent** to the category of Priestley spaces, and therefore also to the category of spectral spaces.*

- For any bounded distributive lattice L, define the dual space $(X_L, \leq, \tau)$:
  - points of X_L are bounded lattice homomorphisms $L \to 2$;
  - $\leq$ is the point-wise order;
  - τ is generated by the family of sets $\widehat{a}$ and $\widehat{a}^c$, as a ranges over L, where:

$$\widehat{a} \stackrel{\text{def}}{=} \{x \in X \mid x(a) = 1\} \text{ and } \widehat{a}^c = \{x \in X \mid x(a) = 0\}$$

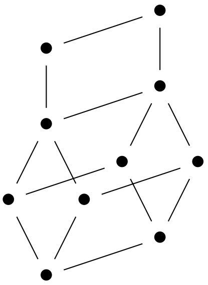
- Then $(X, \leq, \tau)$ is a Priestley space.
- For $h\colon L' \to L$, define $f_h\colon X_L \to X_{L'}$ by $f_h(x) \stackrel{\text{def}}{=} x \circ h$.
- This gives a natural isomorphism $\mathrm{Hom}_{\mathbf{DL}}(L', L) \to \mathrm{Hom}_{\mathbf{Pr}}(X_L, X_{L'})$.

Example 1

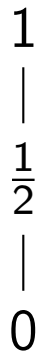


L

Example 1

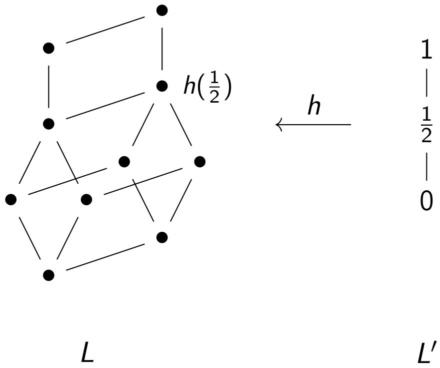


L

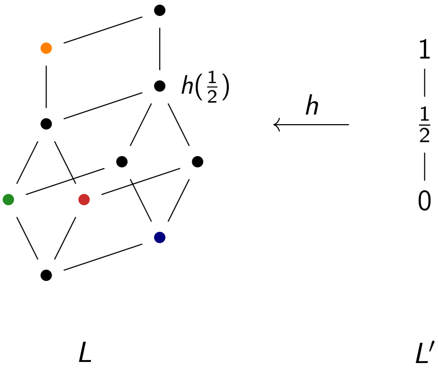


L'

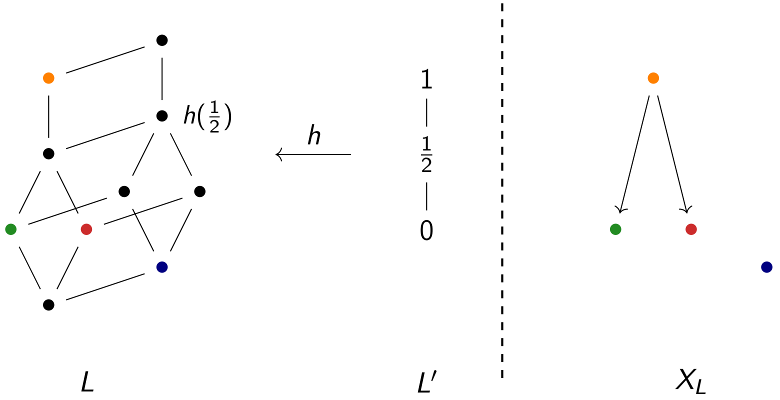
Example 1



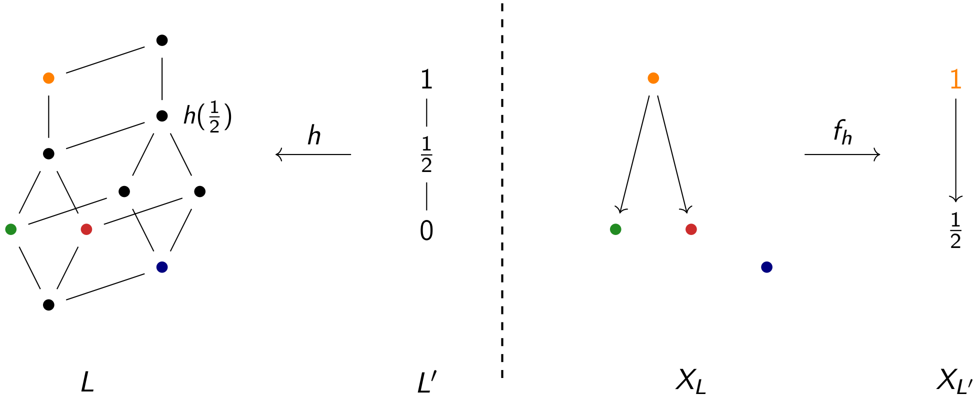
Example 1



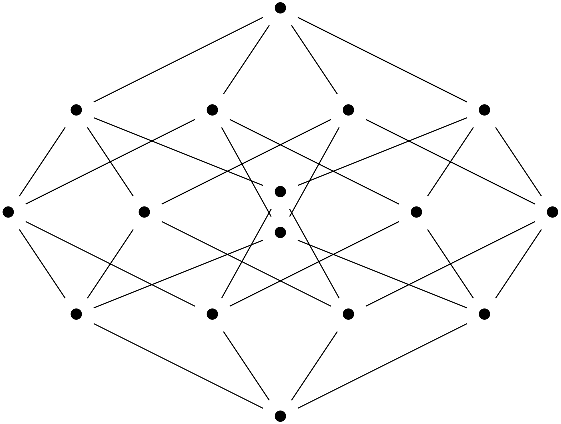
Example 1



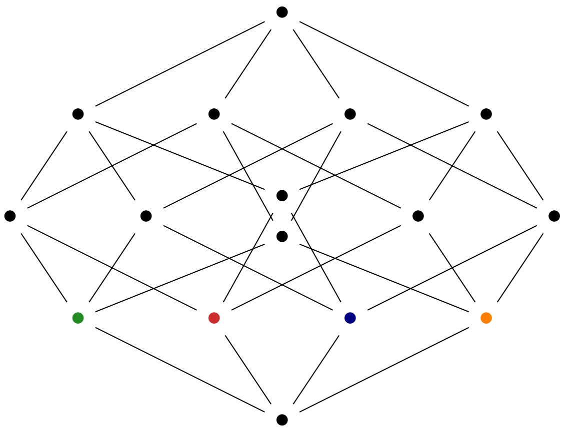
Example 1



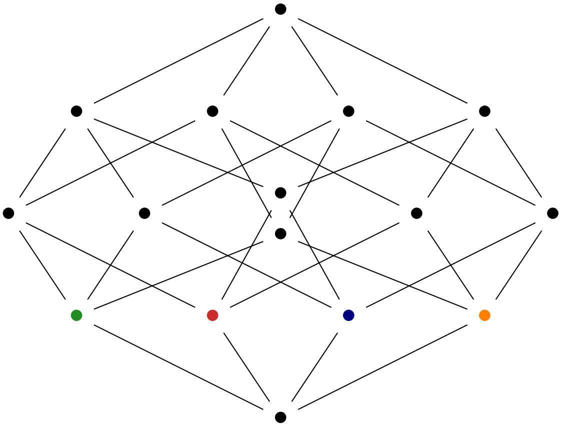
Example 2



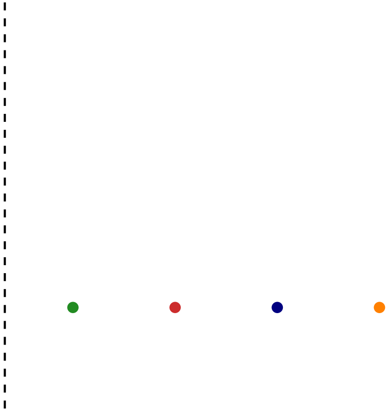
Example 2



Example 2



$\mathcal{P}(\mathbf{4})$



$\mathbf{4}$

Example 3

free Boolean algebra on $\mathbb{N}$

?

Example 3

free Boolean algebra on $\mathbb{N}$

Cantor space $\mathbf{2}^{\mathbb{N}}$

Stone-Priestley duality: A coherent view

- A different way of seeing the above story is using the general framework of frames:

Stone-Priestley duality: A coherent view

- A different way of seeing the above story is using the general framework of frames:
- A bounded distributive lattice is a partial order $(L, \leq)$ such that every **finite** subset $S \subseteq L$ has a least upper bound $\bigvee S$ ('join of S'), and a greatest lower bound $\bigwedge S$ ('meet of S'), and, for any $a \in L$, $a \wedge \bigvee S = \bigvee_{s \in S}(a \wedge s)$.

Stone-Priestley duality: A coherent view

- A different way of seeing the above story is using the general framework of frames:
- A frame is a partial order $(L, \leq)$ such that every subset $S \subseteq L$ has a least upper bound $\bigvee S$ ('join of S'), and a greatest lower bound $\bigwedge S$ ('meet of S'), and, for any $a \in L$, $a \wedge \bigvee S = \bigvee_{s \in S}(a \wedge s)$.

Stone-Priestley duality: A coherent view

- A different way of seeing the above story is using the general framework of frames:
- A frame is a partial order $(L, \leq)$ such that every subset $S \subseteq L$ has a least upper bound $\bigvee S$ ('join of S'), and a greatest lower bound $\bigwedge S$ ('meet of S'), and, for any $a \in L$, $a \wedge \bigvee S = \bigvee_{s \in S}(a \wedge s)$.
- Another version of Stone duality gives a dual equivalence between *spatial* frames and *sober* spaces.

Stone-Priestley duality: A coherent view

- A different way of seeing the above story is using the general framework of frames:
- A frame is a partial order $(L, \leq)$ such that every subset $S \subseteq L$ has a least upper bound $\bigvee S$ ('join of S'), and a greatest lower bound $\bigwedge S$ ('meet of S'), and, for any $a \in L$, $a \wedge \bigvee S = \bigvee_{s \in S}(a \wedge s)$.
- Another version of Stone duality gives a dual equivalence between *spatial* frames and *sober* spaces.
- The category **DL** **embeds** in the category of frames:
  - For any L, we have the *coherent* frame of ideals of L;
  - For any lattice homomorphism $L' \to L$ we have a *compact element preserving* frame homomorphism from the lattice ideals of L' to the lattice ideals of L.

Stone-Priestley duality: A coherent view

- A different way of seeing the above story is using the general framework of frames:
- A frame is a partial order $(L, \leq)$ such that every subset $S \subseteq L$ has a least upper bound $\bigvee S$ ('join of S'), and a greatest lower bound $\bigwedge S$ ('meet of S'), and, for any $a \in L$, $a \wedge \bigvee S = \bigvee_{s \in S}(a \wedge s)$.
- Another version of Stone duality gives a dual equivalence between *spatial* frames and *sober* spaces.
- The category **DL** **embeds** in the category of frames:
  - For any L, we have the *coherent* frame of ideals of L;
  - For any lattice homomorphism $L' \to L$ we have a *compact element preserving* frame homomorphism from the lattice ideals of L' to the lattice ideals of L.
- The duality spatial frames – sober spaces restricts to a dual equivalence between coherent frames and spectral spaces.

Overview

Duality

Domains

Automata

Spectral dcpo's

- A directedly complete partial order (dcpo) is a poset $(X, \leq)$ in which every *directed* subset D has a supremum.

Spectral dcpo's

- A directedly complete partial order (dcpo) is a poset $(X, \leq)$ in which every *directed* subset D has a supremum.
- A subset C of a dcpo $(X, \leq)$ is Scott-closed if it is a down-set closed under directed suprema.

Spectral dcpo's

- A directedly complete partial order (dcpo) is a poset $(X, \leq)$ in which every *directed* subset D has a supremum.
- A subset C of a dcpo $(X, \leq)$ is Scott-closed if it is a down-set closed under directed suprema.
- A spectral dcpo is a spectral space (X, τ) such that τ is equal to the Scott topology of its specialization order.

Spectral dcpo's

- A directedly complete partial order (dcpo) is a poset $(X, \leq)$ in which every *directed* subset D has a supremum.
- A subset C of a dcpo $(X, \leq)$ is Scott-closed if it is a down-set closed under directed suprema.
- A spectral dcpo is a spectral space (X, τ) such that τ is equal to the Scott topology of its specialization order.

Theorem (Announced by M. Erné 2009, see Thm. 7.38 in our book)

A topological space (X, τ) is a spectral dcpo if, and only if, the topology τ is coherent, sober, and has a base of finitely generated open up-sets.

Spectral dcpo's

- A directedly complete partial order (dcpo) is a poset $(X, \leq)$ in which every *directed* subset D has a supremum.
- A subset C of a dcpo $(X, \leq)$ is Scott-closed if it is a down-set closed under directed suprema.
- A spectral dcpo is a spectral space (X, τ) such that τ is equal to the Scott topology of its specialization order.

Theorem (Announced by M. Erné 2009, see Thm. 7.38 in our book)

A topological space (X, τ) is a spectral dcpo if, and only if, the topology τ is coherent, sober, and has a base of finitely generated open up-sets.

- The proof uses a non-constructive choice via *Rudin's lemma*.

Spectral domains

- A domain is a dcpo in which every element is a **directed** supremum of the set of elements that are *way below* it:

Spectral domains

- A domain is a dcpo in which every element is a **directed** supremum of the set of elements that are *way below* it:
  - x is way below y ($x \ll y$) if, for every directed set D such that $y \leq \bigvee D$, there exists $d \in D$ such that $x \leq d$.

Spectral domains

- A domain is a dcpo in which every element is a **directed** supremum of the set of elements that are *way below* it:
  - x is way below y ($x \ll y$) if, for every directed set D such that $y \leq \bigvee D$, there exists $d \in D$ such that $x \leq d$.
  - x is compact if $x \ll x$.

Spectral domains

- A domain is a dcpo in which every element is a **directed** supremum of the set of elements that are *way below* it:
  - x is way below y ($x \ll y$) if, for every directed set D such that $y \leq \bigvee D$, there exists $d \in D$ such that $x \leq d$.
  - x is compact if $x \ll x$.
- A spectral domain is a spectral dcpo which is also a domain.

Spectral domains

- A domain is a dcpo in which every element is a **directed** supremum of the set of elements that are *way below* it:
  - x is way below y ($x \ll y$) if, for every directed set D such that $y \leq \bigvee D$, there exists $d \in D$ such that $x \leq d$.
  - x is compact if $x \ll x$.
- A spectral domain is a spectral dcpo which is also a domain.

Proposition

*The compact-open sets of a spectral domain are exactly the sets of the form $\uparrow F$, with F a **finite** set of compact elements.*

Spectral domains

- A domain is a dcpo in which every element is a **directed** supremum of the set of elements that are *way below* it:
  - x is way below y ($x \ll y$) if, for every directed set D such that $y \leq \bigvee D$, there exists $d \in D$ such that $x \leq d$.
  - x is compact if $x \ll x$.
- A spectral domain is a spectral dcpo which is also a domain.

Proposition

The compact-open sets of a spectral domain are exactly the sets of the form $\uparrow F$, with F a ***finite*** *set of compact elements.*

- Moreover, spectral domains are algebraic: every element is a directed supremum of the **compact** elements way below it.

Spectral domains

- A domain is a dcpo in which every element is a **directed** supremum of the set of elements that are *way below* it:
  - x is way below y ($x \ll y$) if, for every directed set D such that $y \leq \bigvee D$, there exists $d \in D$ such that $x \leq d$.
  - x is compact if $x \ll x$.
- A spectral domain is a spectral dcpo which is also a domain.

Proposition

*The compact-open sets of a spectral domain are exactly the sets of the form $\uparrow F$, with F a **finite** set of compact elements.*

- Moreover, spectral domains are algebraic: every element is a directed supremum of the **compact** elements way below it.
- The set of compact elements $\mathrm{K}(X)$ entirely describes the domain X.

Characterization of spectral domains

- For an algebraic domain X, if we know **only** $\mathrm{K}(X)$, we can determine whether or not X is a spectral domain.

Characterization of spectral domains

- For an algebraic domain X, if we know **only** $\mathrm{K}(X)$, we can determine whether or not X is a spectral domain.
- When F is a subset of a poset P, a minimal upper bound of F is a **minimal** (not necessarily minimum!) element of the set of upper bounds of F.

Characterization of spectral domains

- For an algebraic domain X, if we know **only** $\mathrm{K}(X)$, we can determine whether or not X is a spectral domain.
- When F is a subset of a poset P, a minimal upper bound of F is a **minimal** (not necessarily minimum!) element of the set of upper bounds of F.
- A poset P is finitely mub-complete if, for all finite $F \subseteq P$:

Characterization of spectral domains

- For an algebraic domain X, if we know **only** $\mathrm{K}(X)$, we can determine whether or not X is a spectral domain.
- When F is a subset of a poset P, a minimal upper bound of F is a **minimal** (not necessarily minimum!) element of the set of upper bounds of F.
- A poset P is finitely mub-complete if, for all finite $F \subseteq P$:
  - F has **finitely many** minimal upper bounds;

Characterization of spectral domains

- For an algebraic domain X, if we know **only** $\mathrm{K}(X)$, we can determine whether or not X is a spectral domain.
- When F is a subset of a poset P, a minimal upper bound of F is a **minimal** (not necessarily minimum!) element of the set of upper bounds of F.
- A poset P is finitely mub-complete if, for all finite $F \subseteq P$:
  - F has **finitely many** minimal upper bounds;
  - every upper bound of F has a minimal upper bound below it.

Characterization of spectral domains

- For an algebraic domain X, if we know **only** $\mathrm{K}(X)$, we can determine whether or not X is a spectral domain.
- When F is a subset of a poset P, a minimal upper bound of F is a **minimal** (not necessarily minimum!) element of the set of upper bounds of F.
- A poset P is finitely mub-complete if, for all finite $F \subseteq P$:
  - F has **finitely many** minimal upper bounds;
  - every upper bound of F has a minimal upper bound below it.

Theorem ("2/3 SFP")

*A domain X is a **spectral** domain if, and only if, X is algebraic, the minimal upper bounds of any finite set $F \subseteq \mathrm{K}(X)$ are all in $\mathrm{K}(X)$, and $\mathrm{K}(X)$ is finitely mub-complete.*

Duality for spectral domains

- Spectral spaces are dual to distributive lattices.

Duality for spectral domains

- Spectral spaces are dual to distributive lattices.
- Which distributive lattices are dual to spectral **domains**?

Duality for spectral domains

- Spectral spaces are dual to distributive lattices.

- Which distributive lattices are dual to spectral **domains**?
  - An element p in a bounded distributive lattice L is (finitely) join prime if, for every finite subset F of L, $p \leq \bigvee F$ implies $\uparrow p \cap F \neq \emptyset$.

Duality for spectral domains

- Spectral spaces are dual to distributive lattices.
- Which distributive lattices are dual to spectral **domains**?
  - An element p in a bounded distributive lattice L is (finitely) join prime if, for every finite subset F of L, $p \leq \bigvee F$ implies $\uparrow p \cap F \neq \emptyset$.
  - L has enough join primes if every element is the join of a **finite** set of join primes.

Duality for spectral domains

- Spectral spaces are dual to distributive lattices.

- Which distributive lattices are dual to spectral **domains**?
  - An element p in a bounded distributive lattice L is (finitely) join prime if, for every finite subset F of L, $p \leq \bigvee F$ implies ${\uparrow}p \cap F \neq \emptyset$.
  - L has enough join primes if every element is the join of a **finite** set of join primes.

Theorem

Stone duality restricts to a dual equivalence between bounded distributive lattices with enough join primes and spectral domains.

Priestley spaces as profinite posets

- A finite poset is a Priestley space in the *discrete* topology.

Priestley spaces as profinite posets

- A finite poset is a Priestley space in the *discrete* topology.
- In the category **KOrd**, a *projective* limit of Priestley spaces is a Priestley space.

Priestley spaces as profinite posets

- A finite poset is a Priestley space in the *discrete* topology.
- In the category **KOrd**, a *projective* limit of Priestley spaces is a Priestley space.
  - A projective diagram is a functor $D\colon I \to \mathbf{KOrd}$ with I a downwards directed poset.

Priestley spaces as profinite posets

- A finite poset is a Priestley space in the *discrete* topology.
- In the category **KOrd**, a *projective* limit of Priestley spaces is a Priestley space.
  - A projective diagram is a functor $D\colon I \to \mathbf{KOrd}$ with I a downwards directed poset.
- **Every** Priestley space is a projective limit of finite posets.

Priestley spaces as profinite posets

- A finite poset is a Priestley space in the *discrete* topology.
- In the category **KOrd**, a *projective* limit of Priestley spaces is a Priestley space.
  - A projective diagram is a functor $D\colon I \to \mathbf{KOrd}$ with I a downwards directed poset.
- **Every** Priestley space is a projective limit of finite posets.
- This gives yet another view on Stone-Priestley duality:

$$\mathbf{DL} \simeq \mathit{Ind}(\mathbf{DL}_{\mathrm{Fin}}) \simeq^{\mathrm{op}} \mathit{Pro}(\mathbf{DL}_{\mathrm{Fin}}^{\mathrm{op}}) \simeq \mathit{Pro}(\mathbf{Pos}_{\mathrm{Fin}}) \simeq \mathbf{Pr}$$

Function spaces

- Given domains X and Y in some class $\mathbf{C}$, we would like the *space of Scott-continuous functions* $[X, Y]$ to again lie in $\mathbf{C}$.

Function spaces

- Given domains X and Y in some class $\mathbf{C}$, we would like the *space of Scott-continuous functions* $[X, Y]$ to again lie in $\mathbf{C}$.
  - The topology on $[X, Y]$ is generated by the sets

    $$K \to U \stackrel{\text{def}}{=} \{f \in [X, Y] : f[K] \subseteq U\}$$

    with K compact-saturated in X, and U open in Y.

Function spaces

- Given domains X and Y in some class $\mathbf{C}$, we would like the *space of Scott-continuous functions* $[X, Y]$ to again lie in $\mathbf{C}$.
  - The topology on $[X, Y]$ is generated by the sets

$$K \to U \stackrel{\text{def}}{=} \{f \in [X, Y] : f[K] \subseteq U\}$$

    with K compact-saturated in X, and U open in Y.

- Plotkin: Consider $\mathbf{C}$ = bifinite domains.

Function spaces

- Given domains X and Y in some class $\mathbf{C}$, we would like the *space of Scott-continuous functions* $[X, Y]$ to again lie in $\mathbf{C}$.
  - The topology on $[X, Y]$ is generated by the sets

$$K \to U \stackrel{\text{def}}{=} \{f \in [X, Y] : f[K] \subseteq U\}$$

    with K compact-saturated in X, and U open in Y.

- Plotkin: Consider $\mathbf{C}$ = bifinite domains.

- Abramsky: **Duality** lets us analyze this situation:

Function spaces

- Given domains X and Y in some class $\mathbf{C}$, we would like the *space of Scott-continuous functions* $[X, Y]$ to again lie in $\mathbf{C}$.
  - The topology on $[X, Y]$ is generated by the sets

$$K \to U \stackrel{\text{def}}{=} \{f \in [X, Y] : f[K] \subseteq U\}$$

    with K compact-saturated in X, and U open in Y.

- Plotkin: Consider $\mathbf{C}$ = bifinite domains.
- Abramsky: **Duality** lets us analyze this situation:
  - The construction $(X, Y) \mapsto [X, Y]$, when X and Y are spectral domains, has a natural **dual** construction.

Bifinite spectral spaces

- We view bifinite domains as bifinite **spectral spaces**:

Bifinite spectral spaces

- We view bifinite domains as bifinite **spectral spaces**:
  - An embedding projection pair (EPP) between spectral spaces is an adjoint pair $e\colon X \leftrightarrows Y : p$ of coherent morphisms, with e injective (and hence p surjective).

Bifinite spectral spaces

- We view bifinite domains as bifinite **spectral spaces**:
  - An embedding projection pair (EPP) between spectral spaces is an adjoint pair $e\colon X \leftrightarrows Y : p$ of coherent morphisms, with e injective (and hence p surjective).
  - A spectral space is bifinite if it is the projective limit in **Spec** of the projections of its finite-domain EPP's.

Bifinite spectral spaces

- We view bifinite domains as bifinite **spectral spaces**:
  - An embedding projection pair (EPP) between spectral spaces is an adjoint pair $e\colon X \leftrightarrows Y : p$ of coherent morphisms, with e injective (and hence p surjective).
  - A spectral space is bifinite if it is the projective limit in **Spec** of the projections of its finite-domain EPP's.
- A bifinite domain is **always** a spectral domain.

Bifinite spectral spaces

- We view bifinite domains as bifinite **spectral spaces**:
  - An embedding projection pair (EPP) between spectral spaces is an adjoint pair $e\colon X \leftrightarrows Y : p$ of coherent morphisms, with e injective (and hence p surjective).
  - A spectral space is bifinite if it is the projective limit in **Spec** of the projections of its finite-domain EPP's.
- A bifinite domain is **always** a spectral domain.
- For a spectral domain X, we can also characterize bifiniteness as a property of the poset of compact elements $\mathrm{K}(X)$ or of the lattice of compact-open subsets of X.

Bifinite spectral spaces

- We view bifinite domains as bifinite **spectral spaces**:
  - An embedding projection pair (EPP) between spectral spaces is an adjoint pair $e\colon X \leftrightarrows Y : p$ of coherent morphisms, with e injective (and hence p surjective).
  - A spectral space is bifinite if it is the projective limit in **Spec** of the projections of its finite-domain EPP's.
- A bifinite domain is **always** a spectral domain.
- For a spectral domain X, we can also characterize bifiniteness as a property of the poset of compact elements $\mathrm{K}(X)$ or of the lattice of compact-open subsets of X.
- The interest of duality for bifinite domains is to **solve domain equations**.

The relation space and its dual

- For X a topological space, let $\mathcal{V}^{\uparrow}(X)$ the upper Vietoris space:

The relation space and its dual

- For X a topological space, let $\mathcal{V}^{\uparrow}(X)$ the upper Vietoris space:
  - points of $\mathcal{V}^{\uparrow}(X)$ are compact-saturated subsets of X;

The relation space and its dual

- For X a topological space, let $\mathcal{V}^{\uparrow}(X)$ the upper Vietoris space:
  - points of $\mathcal{V}^{\uparrow}(X)$ are compact-saturated subsets of X;
  - topology on $\mathcal{V}^{\uparrow}(X)$ is generated by, for $U \subseteq X$ open:

$$\Box U \stackrel{\text{def}}{=} \{K \in \mathcal{V}^{\uparrow}(X) \mid K \subseteq U\}.$$

The relation space and its dual

- For X a topological space, let $\mathcal{V}^{\uparrow}(X)$ the upper Vietoris space:
  - points of $\mathcal{V}^{\uparrow}(X)$ are compact-saturated subsets of X;
  - topology on $\mathcal{V}^{\uparrow}(X)$ is generated by, for $U \subseteq X$ open:

$$\Box U \stackrel{\text{def}}{=} \{K \in \mathcal{V}^{\uparrow}(X) \mid K \subseteq U\}.$$

- If Y is spectral, then so is $\mathcal{V}^{\uparrow}(Y)$.

The relation space and its dual

- For X a topological space, let $\mathcal{V}^{\uparrow}(X)$ the upper Vietoris space:
  - points of $\mathcal{V}^{\uparrow}(X)$ are compact-saturated subsets of X;
  - topology on $\mathcal{V}^{\uparrow}(X)$ is generated by, for $U \subseteq X$ open:

$$\Box U \stackrel{\text{def}}{=} \{K \in \mathcal{V}^{\uparrow}(X) \mid K \subseteq U\}.$$

- If Y is spectral, then so is $\mathcal{V}^{\uparrow}(Y)$.
- Let X, Y spectral spaces with dual lattices L, M.
  The space $[X, \mathcal{V}^{\uparrow}(Y)]$ is always spectral, with dual lattice $F_{\rightarrow}(L, M)$:

The relation space and its dual

- For X a topological space, let $\mathcal{V}^{\uparrow}(X)$ the upper Vietoris space:
  - points of $\mathcal{V}^{\uparrow}(X)$ are compact-saturated subsets of X;
  - topology on $\mathcal{V}^{\uparrow}(X)$ is generated by, for $U \subseteq X$ open:

$$\Box U \stackrel{\text{def}}{=} \{K \in \mathcal{V}^{\uparrow}(X) \mid K \subseteq U\}.$$

- If Y is spectral, then so is $\mathcal{V}^{\uparrow}(Y)$.
- Let X, Y spectral spaces with dual lattices L, M.
  The space $[X, \mathcal{V}^{\uparrow}(Y)]$ is **always** spectral, with dual lattice $F_{\rightarrow}(L, M)$:
  - $F_{\rightarrow}(L, M)$ is a quotient of $F_{\mathbf{DL}}(L, M)$. Write $a \rightarrow b$ for elements of $F_{\mathbf{DL}}(L, M)$.

The relation space and its dual

- For X a topological space, let $\mathcal{V}^{\uparrow}(X)$ the upper Vietoris space:
  - points of $\mathcal{V}^{\uparrow}(X)$ are compact-saturated subsets of X;
  - topology on $\mathcal{V}^{\uparrow}(X)$ is generated by, for $U \subseteq X$ open:

$$\Box U \stackrel{\text{def}}{=} \{K \in \mathcal{V}^{\uparrow}(X) \mid K \subseteq U\}.$$

- If Y is spectral, then so is $\mathcal{V}^{\uparrow}(Y)$.
- Let X, Y spectral spaces with dual lattices L, M.
  The space $[X, \mathcal{V}^{\uparrow}(Y)]$ is always spectral, with dual lattice $F_{\rightarrow}(L, M)$:
  - $F_{\rightarrow}(L, M)$ is a quotient of $F_{\mathbf{DL}}(L, M)$. Write $a \rightarrow b$ for elements of $F_{\mathbf{DL}}(L, M)$.
  - $F_{\rightarrow}(L, M) \stackrel{\text{def}}{=} F_{\mathbf{DL}}(L, M)/\theta$, with θ generated by the equations

$$\left(\bigvee A\right) \rightarrow b_0 = \bigwedge_{a \in a}(a \rightarrow b) \quad \text{and} \quad a \rightarrow \left(\bigwedge B\right) = \bigwedge_{b \in B}(a_0 \rightarrow b),$$

for any finite $A \cup \{a_0\} \subseteq L$ and $B \cup \{b_0\} \subseteq M$.

Spectrality of the function space

- The subspace $[X, Y]$ of $[X, \mathcal{V}^{\uparrow}(Y)]$ is **not** always spectral.

Spectrality of the function space

- The subspace $[X, Y]$ of $[X, \mathcal{V}^{\uparrow}(Y)]$ is **not** always spectral.
- When it is, it is **dual** to the largest quotient Q of $F_{\rightarrow}(L, M)$ which preserves joins at primes, by which we mean:
  For any homomorphism $x \colon Q \to \mathbf{2}$, $a \in Q$ with $x(a) = 1$, and any finite subset G of Q, there exists $a' \in Q$ with $x(a') = 1$ and

$$a \to \left(\bigvee G\right) \leq_Q \bigvee \{a' \to g \mid g \in G\}.$$

Spectrality of the function space

- The subspace $[X, Y]$ of $[X, \mathcal{V}^{\uparrow}(Y)]$ is **not** always spectral.
- When it is, it is **dual** to the largest quotient Q of $F_{\rightarrow}(L, M)$ which preserves joins at primes, by which we mean:
  For any homomorphism $x\colon Q \to \mathbf{2}$, $a \in Q$ with $x(a) = 1$, and any finite subset G of Q, there exists $a' \in Q$ with $x(a') = 1$ and

$$a \to \left(\bigvee G\right) \leq_Q \bigvee \{a' \to g \mid g \in G\}.$$

- This lets us show that $[X, Y]$ **is** a spectral space whenever X is a spectral domain.

Spectrality of the function space

- The subspace $[X, Y]$ of $[X, \mathcal{V}^{\uparrow}(Y)]$ is **not** always spectral.
- When it is, it is **dual** to the largest quotient Q of $F_{\rightarrow}(L, M)$ which preserves joins at primes, by which we mean:
  For any homomorphism $x \colon Q \to \mathbf{2}$, $a \in Q$ with $x(a) = 1$, and any finite subset G of Q, there exists $a' \in Q$ with $x(a') = 1$ and

$$a \to \left(\bigvee G \right) \leq_Q \bigvee \{a' \to g \mid g \in G\}.$$

- This lets us show that $[X, Y]$ **is** a spectral space whenever X is a spectral domain.
- One further shows that $[X, Y]$ is bifinite if X and Y are.

Domains in duality-theoretic form

- One may use this theory to construct domains X such that $X \cong [X, X]$.

Domains in duality-theoretic form

- One may use this theory to construct domains X such that $X \cong [X, X]$.
- E.g., starting from the Sierpinski space $\mathbb{S}$, one builds a sequence of EPP's

$$\mathbb{S} \leftrightarrows [\mathbb{S}, \mathbb{S}] \leftrightarrows [[\mathbb{S}, \mathbb{S}], [\mathbb{S}, \mathbb{S}]] \leftrightarrows \cdots$$

Domains in duality-theoretic form

- One may use this theory to construct domains X such that $X \cong [X, X]$.
- E.g., starting from the Sierpinski space $\mathbb{S}$, one builds a sequence of EPP's

$$\mathbb{S} \leftrightarrows [\mathbb{S}, \mathbb{S}] \leftrightarrows [[\mathbb{S}, \mathbb{S}], [\mathbb{S}, \mathbb{S}]] \leftrightarrows \cdots$$

- The dual of $\mathbb{S}$ is the three-element lattice $\mathbf{3}$, and we get a dual sequence

$$\mathbf{3} \leftrightarrows F_{\to}(\mathbf{3}, \mathbf{3}) \leftrightarrows F_{\to}(F_{\to}(\mathbf{3}, \mathbf{3}), F_{\to}(\mathbf{3}, \mathbf{3})) \leftrightarrows \cdots$$

Domains in duality-theoretic form

- One may use this theory to construct domains X such that $X \cong [X, X]$.
- E.g., starting from the Sierpinski space $\mathbb{S}$, one builds a sequence of EPP's

$$\mathbb{S} \leftrightarrows [\mathbb{S}, \mathbb{S}] \leftrightarrows [[\mathbb{S}, \mathbb{S}], [\mathbb{S}, \mathbb{S}]] \leftrightarrows \cdots$$

- The dual of $\mathbb{S}$ is the three-element lattice $\mathbf{3}$, and we get a dual sequence

$$\mathbf{3} \leftrightarrows F_{\rightarrow}(\mathbf{3}, \mathbf{3}) \leftrightarrows F_{\rightarrow}(F_{\rightarrow}(\mathbf{3}, \mathbf{3}), F_{\rightarrow}(\mathbf{3}, \mathbf{3})) \leftrightarrows \cdots$$

- The limit of domains becomes a colimit of lattices, and can be easier to compute, and prove things about.

Overview

Duality

Domains

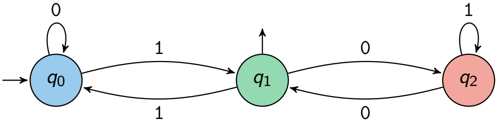
Automata

Automata, monoids, and logic

- A **programming problem**: given a natural number in binary, $w \in \{0,1\}^*$, determine if w is congruent to 1 modulo 3.

Automata, monoids, and logic

- A **programming problem**: given a natural number in binary, $w \in \{0,1\}^*$, determine if w is congruent to 1 modulo 3.
- **Solution 1:** an *automaton* A:

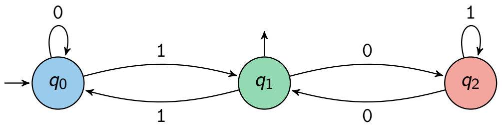


Answer yes iff A accepts w.

Automata, monoids, and logic

- A **programming problem**: given a natural number in binary, $w \in \{0,1\}^*$, determine if w is congruent to 1 modulo 3.

- **Solution 1:** an *automaton* A:



  Answer yes iff A accepts w.

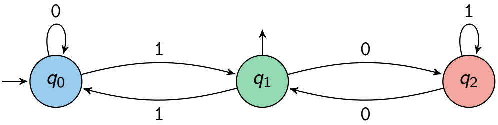
- **Solution 2:** a *homomorphism* $\varphi\colon \{0,1\}^* \to S_3$ defined by

$$0 \mapsto (1\,2), \quad 1 \mapsto (0\,1).$$

  Answer yes iff the permutation $\varphi(w)$ sends 0 to 1.

Automata, monoids, and logic

- A **programming problem**: given a natural number in binary, $w \in \{0,1\}^*$, determine if w is congruent to 1 modulo 3.
- **Solution 1:** an *automaton* A:



Answer yes iff A accepts w.

- **Solution 3:** a *formula* φ describing *accepting runs* of A:

$$\exists Q_0 \exists Q_1 \exists Q_2 (Q_0(\texttt{first}) \wedge Q_1(\texttt{last}) \wedge$$
$$\forall x [0(x) \wedge Q_0(x) \rightarrow Q_0(Sx)] \wedge [1(x) \wedge Q_0(x) \rightarrow Q_1(Sx)] \wedge \dots).$$

Answer yes iff w satisfies the formula φ.

Regular sets

- A set L of finite words is regular if it satisfies the following equivalent conditions:

Regular sets

- A set L of finite words is regular if it satisfies the following equivalent conditions:
  - L is *definable* by a monadic second order sentence,

Regular sets

- A set L of finite words is regular if it satisfies the following equivalent conditions:
  - L is *definable* by a monadic second order sentence,
  - L is *recognizable* by a finite automaton,

Regular sets

- A set L of finite words is regular if it satisfies the following equivalent conditions:
  - L is *definable* by a monadic second order sentence,
  - L is *recognizable* by a finite automaton,
  - L is *saturated* under a finite index monoid congruence on Σ^*,
    i.e., there exists a surjective homomorphism

$$h \colon \Sigma^* \twoheadrightarrow M,$$

    with M a finite monoid, such that, for some $P \subseteq M$,

$$L = h^{-1}(P).$$

Regular sets

- A set L of finite words is regular if it satisfies the following equivalent conditions:
  - L is *definable* by a monadic second order sentence,
  - L is *recognizable* by a finite automaton,
  - L is *saturated* under a finite index monoid congruence on Σ^*,
    i.e., there exists a surjective homomorphism

    $$h \colon \Sigma^* \twoheadrightarrow M,$$

    with M a finite monoid, such that, for some $P \subseteq M$,

    $$L = h^{-1}(P).$$

- Note that the collection of regular sets of Σ-words is a **Boolean algebra**.

The free profinite monoid

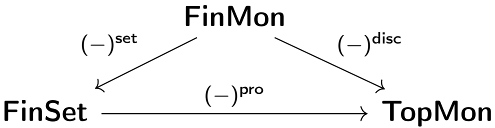
- The free profinite monoid over Σ is the embedding of Σ into a topological monoid Σ^{pro} such that, for every finite monoid M and function $f\colon \Sigma \to M^{\mathsf{set}}$, there exists a unique continuous homomorphism $\overline{f}\colon \Sigma^{\mathsf{pro}} \to M^{\mathsf{disc}}$ that extends f.

The free profinite monoid

- The free profinite monoid over Σ is the embedding of Σ into a topological monoid Σ^{pro} such that, for every finite monoid M and function $f\colon \Sigma \to M^{\mathrm{set}}$, there exists a unique continuous homomorphism $\overline{f}\colon \Sigma^{\mathrm{pro}} \to M^{\mathrm{disc}}$ that extends f.

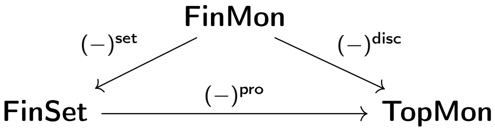
The free profinite monoid

- The free profinite monoid over Σ is the embedding of Σ into a topological monoid Σ^{pro} such that, for every finite monoid M and function $f\colon \Sigma \to M^{\mathsf{set}}$, there exists a unique continuous homomorphism $\overline{f}\colon \Sigma^{\mathsf{pro}} \to M^{\mathsf{disc}}$ that extends f.



The free profinite monoid

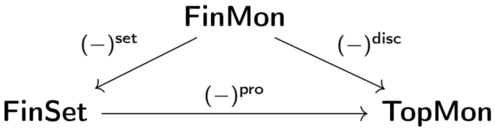
- The free profinite monoid over Σ is the embedding of Σ into a topological monoid Σ^{pro} such that, for every finite monoid M and function $f\colon \Sigma \to M^{\text{set}}$, there exists a unique continuous homomorphism $\overline{f}\colon \Sigma^{\text{pro}} \to M^{\text{disc}}$ that extends f.



$$\frac{f\colon \Sigma \to M^{\text{set}}}{\overline{f}\colon \Sigma^{\text{pro}} \to M^{\text{disc}}}$$

The free profinite monoid

- The free profinite monoid over Σ is the embedding of Σ into a topological monoid Σ^{pro} such that, for every finite monoid M and function $f : \Sigma \to M^{\text{set}}$, there exists a unique continuous homomorphism $\overline{f} : \Sigma^{\text{pro}} \to M^{\text{disc}}$ that extends f.



$$\frac{f : \Sigma \to M^{\text{set}}}{\overline{f} : \Sigma^{\text{pro}} \to M^{\text{disc}}}$$

- Elements of Σ^{pro} are called profinite words over Σ.

Duality between regular sets and free profinite monoid

Theorem. The topological space underlying the free profinite monoid Σ^{pro} is homeomorphic to the Stone dual space of the Boolean algebra of regular sets.

Duality between regular sets and free profinite monid

Theorem. The topological space underlying the free profinite monoid Σ^{pro} is homeomorphic to the Stone dual space of the Boolean algebra of regular sets.

- What is the **monoid** structure on Σ^{pro} dual to?

Duality between regular sets and free profinite monid

Theorem. The topological space underlying the free profinite monoid Σ^{pro} is homeomorphic to the Stone dual space of the Boolean algebra of regular sets.

- What is the **monoid** structure on Σ^{pro} dual to?
- On the Boolean algebra of regular sets, we have the operation, for K, L regular,

$$K \backslash L \stackrel{\text{def}}{=} \{ w \in \Sigma^* \mid \text{ for all } u \in K, uw \in L \}.$$

Duality between regular sets and free profinite monid

Theorem. The topological space underlying the free profinite monoid Σ^{pro} is homeomorphic to the Stone dual space of the Boolean algebra of regular sets.

- What is the **monoid** structure on Σ^{pro} dual to?
- On the Boolean algebra of regular sets, we have the operation, for K, L regular,

$$K \backslash L \stackrel{\text{def}}{=} \{w \in \Sigma^* \mid \text{ for all } u \in K, uw \in L\}.$$

- This is part of a **residuation** structure $(\mathrm{Reg}(\Sigma^*), \backslash, /)$.

Duality between regular sets and free profinite monid

Theorem. The topological space underlying the free profinite monoid Σ^{pro} is homeomorphic to the Stone dual space of the Boolean algebra of regular sets.

- What is the **monoid** structure on Σ^{pro} dual to?
- On the Boolean algebra of regular sets, we have the operation, for K, L regular,

$$K \backslash L \stackrel{\text{def}}{=} \{w \in \Sigma^* \mid \text{ for all } u \in K, uw \in L\}.$$

- This is part of a **residuation** structure $(\mathrm{Reg}(\Sigma^*), \backslash, /)$.
- Its dual is the monoid operation on Σ^{pro}.

Preserving joins at primes, again!

- More generally: **Any** continuous binary operation $\star$ on a Boolean space X comes from a residuation structure on the Boolean algebra of clopen sets.

Preserving joins at primes, again!

- More generally: **Any** continuous binary operation $\star$ on a Boolean space X comes from a residuation structure on the Boolean algebra of clopen sets.

- An operation $\backslash \colon B \times B \to B$ on a Boolean algebra B preserves joins at primes if: For any homomorphism $x \colon B \to 2$, $a \in B$ with $x(a) = 1$, and finite subset G of B, there exists $a' \in B$ with $x(a') = 1$ and

$$a \backslash \left(\bigvee G \right) \leq \bigvee \{ a' \backslash g \mid g \in G \}.$$

Preserving joins at primes, again!

- More generally: **Any** continuous binary operation $\star$ on a Boolean space X comes from a residuation structure on the Boolean algebra of clopen sets.
- An operation $\backslash \colon B \times B \to B$ on a Boolean algebra B preserves joins at primes if: For any homomorphism $x \colon B \to 2$, $a \in B$ with $x(a) = 1$, and finite subset G of B, there exists $a' \in B$ with $x(a') = 1$ and

$$a \backslash \left(\bigvee G\right) \leq \bigvee \{a' \backslash g \mid g \in G\}.$$

Theorem

A Boolean residuation algebra $(B, \backslash, /)$ is dual to a binary topological algebra $(X, \star)$ if, and only if, both $\backslash$ and $/$ preserve joins at primes.

Automata in duality-theoretic form

- Automata theory considers questions of the form: How complex is it to compute a given regular set? For example, given a regular set L,

Automata in duality-theoretic form

- Automata theory considers questions of the form: How complex is it to compute a given regular set? For example, given a regular set L,
  - can it be *defined* with only first-order quantifiers?

Automata in duality-theoretic form

- Automata theory considers questions of the form: How complex is it to compute a given regular set? For example, given a regular set L,
  - can it be *defined* with only first-order quantifiers?
  - can it be *recognized* with monoids that don't have subgroups?

Automata in duality-theoretic form

- Automata theory considers questions of the form: How complex is it to compute a given regular set? For example, given a regular set L,
  - can it be *defined* with only first-order quantifiers?
  - can it be *recognized* with monoids that don't have subgroups?
- We want an **algorithm** that answers this, given as input an automaton for L.

Automata in duality-theoretic form

- Automata theory considers questions of the form: How complex is it to compute a given regular set? For example, given a regular set L,
  - can it be *defined* with only first-order quantifiers?
  - can it be *recognized* with monoids that don't have subgroups?
- We want an **algorithm** that answers this, given as input an automaton for L.
- One may use duality to analyze such questions:

$$\text{subalgebra } F \hookrightarrow \mathrm{Reg}(\Sigma^*) \quad \textit{is dual to} \quad \text{quotient } \Sigma^{\mathrm{pro}} \twoheadrightarrow Q_F$$

Automata in duality-theoretic form

- Automata theory considers questions of the form: How complex is it to compute a given regular set? For example, given a regular set L,
  - can it be *defined* with only first-order quantifiers?
  - can it be *recognized* with monoids that don't have subgroups?
- We want an **algorithm** that answers this, given as input an automaton for L.
- One may use duality to analyze such questions:

$$\text{subalgebra } F \hookrightarrow \mathrm{Reg}(\Sigma^*) \quad \textit{is dual to} \quad \text{quotient } \Sigma^{\mathrm{pro}} \twoheadrightarrow Q_F$$

- For example,

$$FO(\Sigma^*) \hookrightarrow \mathrm{Reg}(\Sigma^*) \quad \textit{is dual to} \quad \Sigma^{\mathrm{pro}} \twoheadrightarrow \Sigma^{\mathrm{pro}}/(x^\omega = x^{\omega+1})$$

Automata in duality-theoretic form

- Automata theory considers questions of the form: How complex is it to compute a given regular set? For example, given a regular set L,
  - can it be *defined* with only first-order quantifiers?
  - can it be *recognized* with monoids that don't have subgroups?
- We want an **algorithm** that answers this, given as input an automaton for L.
- One may use duality to analyze such questions:

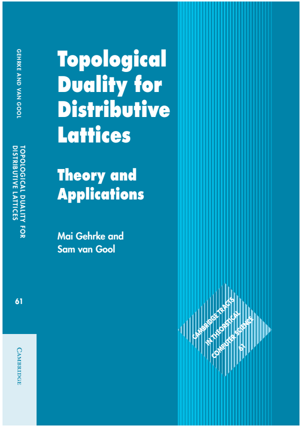
$$\text{subalgebra } F \hookrightarrow \text{Reg}(\Sigma^*) \quad \textit{is dual to} \quad \text{quotient } \Sigma^{\text{pro}} \twoheadrightarrow Q_F$$

- For example,

$$FO(\Sigma^*) \hookrightarrow \text{Reg}(\Sigma^*) \quad \textit{is dual to} \quad \Sigma^{\text{pro}} \twoheadrightarrow \Sigma^{\text{pro}}/(x^\omega = x^{\omega+1})$$

- More about this in Chapter 8 of ...

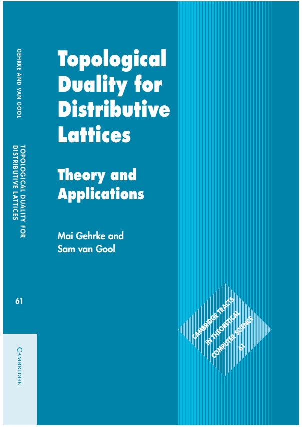
The book



- Potential uses:

Topological Duality for Distributive Lattices: Theory and Applications, by Mai Gehrke and Sam van Gool. Cambridge University Press, 369pp (2024).

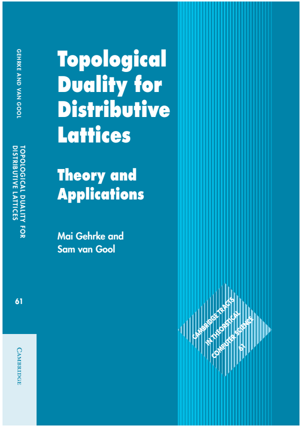
The book



- Potential uses:
  - Advanced undergraduate course on Birkhoff-Priestley duality, covering the first chapters and some methods.

Topological Duality for Distributive Lattices: Theory and Applications, by Mai Gehrke and Sam van Gool. Cambridge University Press, 369pp (2024).

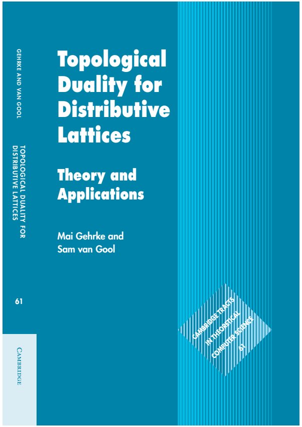
The book



- Potential uses:
  - Advanced undergraduate course on Birkhoff-Priestley duality, covering the first chapters and some methods.
  - Graduate course covering most of the book, possibly focusing on one of the two application chapters.

Topological Duality for Distributive Lattices: Theory and Applications, by Mai Gehrke and Sam van Gool. Cambridge University Press, 369pp (2024).

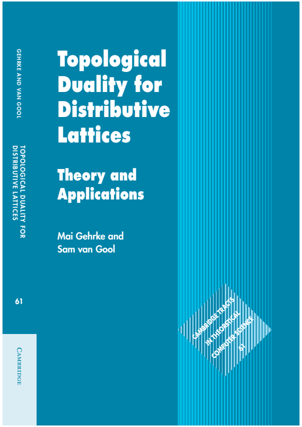
The book



- Potential uses:
  - Advanced undergraduate course on Birkhoff-Priestley duality, covering the first chapters and some methods.
  - Graduate course covering most of the book, possibly focusing on one of the two application chapters.
  - Research monograph: Last two chapters contain new material, which we believe could spurn new research.

Topological Duality for Distributive Lattices: Theory and Applications, by Mai Gehrke and Sam van Gool. Cambridge University Press, 369pp (2024).

The book



Hardcover and e-book.

20% discount code: **TDDL2024**

`www.cambridge.org/9781009349697`

Topological Duality for Distributive Lattices: Theory and Applications,
by Mai Gehrke and Sam van Gool. Cambridge University Press, 369pp (2024).

Thank you.

谢谢你