Stone duality and its formalization

Sam van Gool

IRIF, Université Paris Cité

Lean for the curious mathematician / Lean pour mathématiciens
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About this talk

This talk has a dual aim:

- to provide a mathematical overview of Stone duality theory,

  and

- to invite collaboration on its formalization.

Topological spaces and frames

Duality between points and opens

A point x of a topological space X determines a collection of open neighborhoods,

$$\varepsilon(x) \stackrel{\text{def}}{=} \{U \in \mathcal{O}(X) \mid x \in U\} .$$

The function ε maps X to its 'double dual'.

But what is the 'dual' of a topological space?

This talk will consider two possible, closely related, answers.

Frames

A partial order $(L, \leq)$ is called a complete lattice if any subset S has a supremum, $\bigvee S$, and an infimum, $\bigwedge S$. (Assuming one of the two suffices.)

A frame is a complete lattice $(L, \leq)$ in which finite meets distribute over arbitrary joins, i.e.,

$$u \wedge \left(\bigvee S\right) = \bigvee_{v \in S} (u \wedge v)$$

for any $u \in L$ and $S \subseteq L$.

Examples of frames

- The open sets $\mathcal{O}(X)$ of any topological space X:

$$\bigvee_{i \in I} U_i = \bigcup_{i \in I} U_i \ , \quad U \wedge V = U \cap V \ .$$

- The radical ideals $\mathrm{RId}(R)$ of any ring R:

$$\bigvee_{i \in I} J_i = \sqrt{\bigoplus_{i \in I} J_i} \ , \quad I \wedge J = I \cap J \ .$$

Used in a *very* recent formalization of schemes in Cubical Agda by Zeuner & Hutzler, arXiv:2403.13088v1.

- The regular open subsets of any compact Hausdorff space:

$$\bigvee_{i \in I} R_i = \overline{\bigcup_{i \in I} R_i}^{\circ} \ , \quad R \wedge S = R \cap S \ .$$

Continuous functions and frame homomorphisms

A frame homomorphism is a function preserving $\bigvee$, $\wedge$, and $\top$.

A continuous map between topological spaces

$$f\colon X \to Y$$

gives a frame homomorphism

$$f^{-1}\colon \mathcal{O}(Y) \to \mathcal{O}(X)\ .$$

Categorically: we have defined a functor $\mathcal{O}\colon \mathbf{Top} \to \mathbf{Frm}^{\mathrm{op}}$.

Some people (and some formalization libraries) like to call objects of $\mathbf{Frm}^{\mathrm{op}}$ locales.

Inverting $\mathcal{O}$

Consider a space X and write $L \stackrel{\text{def}}{=} \mathcal{O}(X)$.
Can you reconstruct the space X if you only remember L?

The open-neighborhood function $\varepsilon\colon X \to \mathcal{P}(\mathcal{O}(X))$ gives a way to interpret points of X as subsets of $\mathcal{O}(X)$. More precisely, $\varepsilon(x)$ is always a *completely prime filter* of $\mathcal{O}(X)$. (We will see a definition shortly.)

Facts/Definitions.

1. ε is injective if, and only if, X is T_0.
2. ε is surjective onto the set of completely prime filters if, and only if, X is quasi-sober.
3. A space is called sober if it is T_0 and quasi-sober, i.e., if ε is bijective.

Let us take stock, or: Faisons le point

We have defined a functor $\mathcal{O}\colon \mathbf{Top} \to \mathbf{Frm}^{\mathrm{op}}$, and a way to recover the set X from $\mathcal{O}(X)$, if X is sober.

A few natural questions:

1. How to recover the topology on X?
2. How to recover continuous maps between spaces?
3. Can any frame be reached by the functor $\mathcal{O}$?

The set of points of a frame

Let L be a frame. A homomorphism

$$x \colon L \to \mathbf{2}$$

to the two-element frame $\mathbf{2} = \mathcal{O}(*) = \{0, 1\}$ is called a point of L, and $\mathrm{pt}\, L$ is the set of points of L.

The same people, and libraries, who call objects of $\mathbf{Frm}^{\mathrm{op}}$ *locales* typically denote $\mathbf{2}$ by $\mathbf{1}$.

A completely prime filter is a subset F of L such that $F = x^{-1}(1)$ for some $x \in \mathrm{pt}\, L$.

The points functor

The set of points of L, $\mathrm{pt}\, L$, carries a topology

$$\{\eta(u) \ : \ u \in L\}$$

where

$$\eta(u) \stackrel{\mathrm{def}}{=} \{x \in \mathrm{pt}\, L \ \mid \ x(u) = 1\} \ .$$

Any frame homomorphism $f \colon L \to M$ gives a dual function

$$f^* \colon \ \mathrm{pt}\, M \to \mathrm{pt}\, L \ , \quad x \mapsto x \circ f \ ,$$

which is continuous.

Examples of dual spaces

- Points of $\mathcal{O}X$ correspond to irreducible closed sets of X:

$$x \colon \mathcal{O}X \to \mathbf{2} \quad \longleftrightarrow \quad C_x \stackrel{\text{def}}{=} X \setminus \left(\bigcup \{U \mid x(U) = 0\} \right).$$

- Points of $\mathrm{RId}R$ correspond to prime ideals of R:

$$x \colon \mathrm{RId}R \to \mathbf{2} \quad \longleftrightarrow \quad I_x \stackrel{\text{def}}{=} \bigcup \{J \in \mathrm{RId}R \mid x(J) = 1\}.$$

- Points of $\mathcal{RO}X$ are … there may not be any! E.g. $\mathcal{RO}[0,1]$.

  A point x of a Boolean frame gives an atom $a_x \stackrel{\text{def}}{=} \bigwedge x^{-1}(1)$.

A dual adjunction

Theorem (Stone 1936, Strauss 1958, . . .)

The functors

$$\mathcal{O} \colon \mathbf{Top} \leftrightarrows \mathbf{Frm}^{\mathrm{op}} \colon \mathrm{pt}$$

are an adjunction, with unit and co-unit

$$\varepsilon_X \colon X \to \mathrm{pt}\,\mathcal{O}X \qquad \textit{and} \qquad \eta_L \colon L \to \mathcal{O}\,\mathrm{pt}\,L\ .$$

Formalization: Started at Banff in May 2023,
in Mathlib in October 2023 (PR #4593).

We saw that ε_X is bijective if, and only if, X is sober.
In that case, it is automatically a homeomorphism.
What about η_L?

Spatial frames

For any frame L, the function $\eta_L\colon L \to \mathcal{O}\,\mathrm{pt}\,L$ is surjective (by definition).

L is called spatial if $\eta_L\colon L \to \mathcal{O}\,\mathrm{pt}\,L$ is injective.

Theorem (Stone 1936, Strauss 1958, ...)

The adjunction $\mathcal{O}\colon \textbf{Top} \leftrightarrows \textbf{Frm}^{\mathrm{op}} : \mathrm{pt}$ *cuts down to an equivalence between* **SoberTop** *and* $\textbf{SpatialFrm}^{\mathrm{op}}$.

Not yet in Mathlib. $\to$ Project?

The (pro)finite setting and coherence

Finite frames

A finite frame is the same thing as a (finite) distributive lattice.

Restricted to this setting, the above duality gives:

Theorem (Birkhoff)

Any finite distributive lattice L is isomorphic to

$$\mathcal{U}(P) \stackrel{\text{def}}{=} \{U \;:\; U \textit{ is an upper set}\}\;,$$

where P is the partially ordered set of join-prime elements of L. Monotone functions between posets $P \to Q$ are in bijection with complete lattice homomorphisms $\mathcal{U}(Q) \to \mathcal{U}(P)$.

A start is in Mathlib (Y. Dillies + recent PR by F. Nuccio & me).
$\to$ Project?

Boolean algebras

A Boolean algebra is a distributive lattice L with *complements*, i.e., for any $x \in L$, there is $\neg x \in L$ with $x \vee \neg x = \top$ and $x \wedge \neg x = \bot$.

(Or: a ring in which $x^2 = x$ for all x.)

Corollary (Birkhoff)

Any finite Boolean algebra B is isomorphic to $\mathcal{P}(X)$, where X is the set of atoms of B.

Stone duality for Boolean algebras

Stone's original idea was to extend Birkhoff duality to the infinite:

Theorem (Stone 1937)

Any Boolean algebra B embeds into $\mathcal{P}(X)$, where X is the set of lattice homomorphisms $B \to \mathbf{2}$.

The topology generated on X by the image of B is compact, T_2, and zero-dimensional, and all such spaces arise in this way.

If B is a frame, then any point of B is a lattice homomorphism, but the converse is false.

Stone (1936) proved an analogous, but less well-known, theorem for distributive lattices, see below.

Profinite sets

$=$ a modern categorical view on Stone's dual spaces for BA's.

For a set S, write DS for the discrete topological space on S.

A profinite set is any topological space that is a cofiltered limit of objects DF with F a finite set.

Proposition

The following are equivalent for a topological space X:

1. *X is a profinite set;*
2. *X is compact, T_2 and zero-dimensional;*
3. *X is compact and totally separated, that is, for any $x, y \in X$,*

   if $x \neq y$ then there is clopen $K \subseteq X$ such that $x \in K$ and $y \notin K$.

leanprover-community.github.io/mathlib4_docs/Mathlib/Topology/Category/Profinite/Basic.html

ntation Mathlib.Topology.Category.Profinite.Basic

The category of Profinite Types

We construct the category of profinite topological spaces, often called profinite sets -- perhaps they could be called profinite types in Lean.

The type of profinite topological spaces is called `Profinite`. It has a category instance and is a fully faithful subcategory of `TopCat`. The fully faithful functor is called `Profinite.toTop`.

Implementation notes

A profinite type is defined to be a topological space which is compact, Hausdorff and totally disconnected.

TODO

- Define procategories and prove that `Profinite` is equivalent to `Pro (FintypeCat)`.

Tags

profinite

```
structure Profinite
      :
  Type (u_1 + 1)
```

source

The type of profinite topological spaces.

```
toCompHaus : CompHaus
```

The underlying compact Hausdorff space of a profinite space.

Profinite sets in Mathlib

"**Implementation notes**

A profinite type is defined to be a topological space which is compact, Hausdorff and totally disconnected.

TODO

Define procategories and prove that `Profinite` is equivalent to `Pro(FintypeCat)`."

– Mathlib (on 24 March 2024)

There is some work to do on this interface.

A First Mile-Stone™

Formalize a proof that the following categories are equivalent:

1. Compact totally separated topological spaces;
2. Cofiltered limits in **Top** of objects DF with F finite;
3. Finite-limit-preserving functors **FinSet** $\to$ **Set**;
4. The Pro-completion of **FinSet**.

Notes.

$1 \iff 2$ is essentially in Mathlib already but needs to be stated.

$1 \iff 4$ is more or less in `lean-liquid` and `lean-solid`.

$3 \iff 4$ is 'just category theory' (famous last words).

Formalizing Stone duality for Boolean algebras

Theorem (Stone 1937)

$$\mathbf{BA}^{\mathrm{op}} \simeq \mathbf{Pro}(\mathbf{FinSet}) \ .$$

Proof. Given the First Mile-Stone™, this is easy:

- $\mathbf{FinBA}^{\mathrm{op}} \simeq \mathbf{FinSet}$,
- $\mathbf{Ind}(\mathbf{FinBA}) \simeq \mathbf{BA}$,
- $\mathbf{Ind}(\mathbf{C})^{\mathrm{op}} \simeq \mathbf{Pro}(\mathbf{C}^{\mathrm{op}})$.

... but what can we do until the First Mile-Stone is in Mathlib?

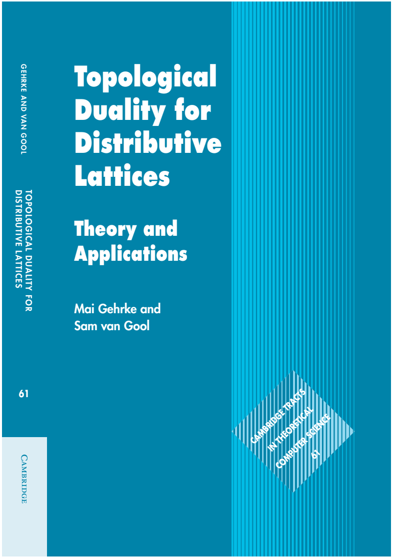
A challenge from Mathlib

Project

Prove the Stone duality theorem that `Profinite` is equivalent to the opposite category of boolean algebras. Then the property of being light says precisely that the corresponding boolean algebra is countable. Maybe constructions of limits and colimits in `LightProfinite` become easier when transporting over this equivalence.

– D. Ásgeirsson in a recent PR (Feb. 2024)

Motivated by this, we had a look at a recent book to see how the Stone duality theorem is proved there...



Gehrke & G., *Topological duality for distributive lattices: Theory and Applications*. Cambridge University Press (2024).

A challenge from Mathlib

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Prove the Stone duality theorem that `Profinite` is equivalent to the opposite category of boolean algebras. Then the property of being light says precisely that the corresponding boolean algebra is countable. Maybe constructions of limits and colimits in `LightProfinite` become easier when transporting over this equivalence.

A recent PR by D. Ásgeirsson (Feb. 2024)

Motivated by this, we ~~had a look at a recent book~~ wrote a detailed proof in LaTeX and started formalizing it (joint work with D. Ásgeirsson and F. Nuccio).

Demo

See the file StoneDuality/BooleanDuality.lean in the LFTCM2024 repository.

$\rightarrow$ Project: Fill in the `sorry`'s.

(Methodological point: Making a project with a lot of 'sorry' can feel a bit icky to a mathematician, but can be a useful way to work on formalization.)

Stone duality for distributive lattices

Theorem (Stone 1936)

$$\mathbf{DL}^{\mathrm{op}} \simeq \mathbf{Pro}(\mathbf{FinT_0}) \ .$$

Proof. Given the Second Mile-Stone™, this is easy:

- $\mathbf{FinDL}^{\mathrm{op}} \simeq \mathbf{FinT_0}$,
- $\mathbf{Ind}(\mathbf{FinDL}) \simeq \mathbf{DL}$.
- $\mathbf{Ind}(\mathbf{C})^{\mathrm{op}} \simeq \mathbf{Pro}(\mathbf{C}^{\mathrm{op}})$.

What is $\mathbf{Pro}\,\mathbf{FinT_0}$?

Spectrality and Coherence

Proposition

A topological space X is a projective limit of finite T_0 spaces if, and only if, it is spectral, that is, compact, sober, and has a basis of compact-open sets which is closed under finite intersections.

Proposition

A space X is spectral if, and only if, the frame $\mathcal{O}(X)$ is coherent, that is, its compact elements are a $\bigvee$-dense bounded sublattice.

An element u in a frame L is compact if for any $S \subseteq L$, $u \leq \bigvee S$ implies $u \leq \bigvee F$ for some finite $F \subseteq S$.

Examples of spectral spaces

- Any finite T_0-space.
- The Zariski spectrum of any ring R. The associated distributive lattice consists of the finitely generated radical ideals of R.

Theorem (Hochster 1969)

Every spectral space is the Zariski spectrum of some ring.

Proof. Interesting. □

Corollary

Every finite distributive lattice is the lattice of finitely generated radical ideals of some ring R.

Formalizing even this corollary is probably tough. Non-trivial paper proof.

The category of spectral spaces

A spectral space X is a projective limit of finite T_0-spaces.

However: not every continuous function $X \to Y$ between spectral spaces factors through the limit diagram!

A function $f\colon X \to Y$ between spectral spaces is called coherent if $f^{-1}(K)$ is compact-open for any compact-open set $K \subseteq Y$.

Taking stock: Stone's dualities

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\mathbf{BoolAlg} \arrow[r, bend right] \arrow[d] \arrow[r, phantom, "\simeq^{\mathrm{op}}"] & \mathbf{ProFinSet} \arrow[l, bend right] \arrow[d] \\
\mathbf{DistLat} \arrow[r, bend right] \arrow[d, "\text{not full!}"] \arrow[r, phantom, "\simeq^{\mathrm{op}}"] & \mathbf{Spec}_{coh} \arrow[l, bend right] \arrow[d, "\text{not full!}"] \\
\mathbf{Frm} \arrow[r, bend right, "\mathrm{pt}"'] \arrow[r, phantom, "\simeq^{\mathrm{op}}"] & \mathbf{Top}_{cts} \arrow[l, bend right, "\mathcal{O}"']
\end{tikzcd}
$$

Distributive lattices and Ordered spaces

Patching up the spectral topology

For a topological space X and $x, y \in X$, the specialization order is

$$x \rightsquigarrow y \iff y \in \mathrm{cl}(\{x\}) \ .$$

Any spectral topology σ on a set X has an inverse topology σ^{∂}, which is also spectral, and has the inverse specialization order.

The patch topology σ^{p} is the smallest containing both σ and σ^{∂}.

Proposition

The partially ordered topological space $(X, \sigma^{p}, \rightsquigarrow)$ is compact and totally order-separated: for any $x, y \in X$, if $x \not\lesssim y$, then there is a clopen upper set $K \subseteq X$ such that $x \in K$ and $y \notin K$.

Such a structure is now called a Priestley space, after her Ph.D. work (1970).

Spectral and Priestley

Let $(X, \pi, \leq)$ a Priestley space. The topology of open upper sets is spectral, with inverse the topology of open lower sets.

Proposition

$\mathbf{Spec}_{coh}$ *is isomorphic to the category of Priestley spaces with continuous monotone maps.*

$\rightarrow$ Possible Project

The Hausdorff spectral spaces ($=$ profinite sets) correspond to the Priestley spaces with trivial specialization order.

Profinite posets

As with profinite sets, there is a fully faithful functor

$$D\colon \mathbf{FinPoset} \to \mathbf{Priestley}$$

which maps a finite poset $(P, \leq)$ to $(P, \tau_{\mathrm{discrete}}, \leq)$.

Proposition

The category of Priestley spaces is equivalent to the Pro-completion of **FinPoset**.

Priestley duality

There is a dual equivalence of categories

$$\mathrm{Spec}\colon \mathbf{DL} \leftrightarrows \mathbf{Priestley}^{\mathrm{op}} : \mathrm{Clp}^{\uparrow} \,,$$

where $\mathrm{Spec}\, L$ is the space of lattice points of L and $\mathrm{Clp}^{\uparrow}X$ is the lattice of clopen upper sets of X.

The frame of open upper sets of $\mathrm{Spec}\, L$ is isomorphic to $\mathrm{Idl}(L)$.

An open mapping theorem and interpolation

Heyting algebras

Remember frames: complete lattices satisfying the law

$$u \wedge \left(\bigvee S\right) = \bigvee_{v \in S} (u \wedge v)$$

Any frame L admits an operation $\Rightarrow$, defined as the residual of $\wedge$:

$$u \Rightarrow v \stackrel{\mathrm{def}}{=} \bigvee \{x \in L \mid u \wedge x \leq v\}$$

For example, when $L = \mathcal{O}(X)$: $U \Rightarrow V = \mathrm{int}((X \setminus U) \cup V)$.

A Heyting algebra is a distributive lattice in which $- \Rightarrow -$ exists.

Frame $=$ Complete Heyting algebra (but careful with morphisms!)

Heyting algebras, dually

Theorem (Esakia 1974)

A distributive lattice L is a Heyting algebra if, and only if, in its Priestley space $\mathrm{Spec}\, L$, the upper set generated by an open set is always open.

Such Priestley spaces are called Esakia spaces.

An open mapping theorem

Inspired by work of Pitts, Ghilardi & Zawadowski on a property of intuitionistic logic called uniform interpolation, we proved:

Theorem (G. & Reggio 2018)

Any continuous bounded map between finitely copresented Esakia spaces is an open map.

Via duality, this gives a different, semantic, proof of:

Theorem (Pitts 1992)

Any homomorphism between finitely presented Heyting algebras has left and right adjoints.

Uniform Interpolant Calculator: an example of extraction

We recently implemented Pitts' original construction in Coq.
It enabled us to apply the construction also to other contexts.

Coq lets you extract verified code into the language OCaml.

We transpiled the OCaml program to a Javascript application that can be run in the browser.

Férée, G., van der Giessen, Shillito (2024)

A quote from Stone

Perhaps [the discovery of Stone duality] and other things that have happened in the course of my research suggest that in many kinds of mathematical work the key is asking the 'right' questions. Once the question is posed the answer becomes a matter of persistent analysis. Of course, the big 'unsolved' problems (Fermat theorem, Riemann hypothesis, etc.) may provide counterexamples. Still many problems seem to become easier when they can be twisted somehow into new forms converting them into 'right' questions.

– M. H. Stone, "A reminiscence on the extension of the Weierstrass approximation theorem" (1976).

An updated version?

Perhaps [the formalization of Stone duality] and other things that have happened in the course of my research suggest that in many kinds of mathematical work the key is formalizing the 'right' `sorry`'s. Once the `sorry` is posted the answer becomes a matter of persistent analysis. Of course, the big 'unsolved' problems (Fermat theorem, Riemann hypothesis, etc.) may provide counterexamples. Still many problems seem to become easier when they can be twisted somehow into new forms converting them into 'right' `sorry's`.

Thank you for your attention.

Appendix: Overview of possible project ideas

1. Cut down the sober-spatial equivalence (p. 13)
2. Complete the proof of Birkhoff's theorem (p. 14)
3. The various equivalent definitions of profinite types (p. 18)
4. Work towards the Mile-Stones (pp. 19 and 25)
5. For the brave: (Finite) Hochster's theorem (p. 27)
6. Priestley duality (p. 31)
7. Open mapping theorem and semantic interpolation (p. 36)
8. Any other ideas you might have gotten from this presentation (come talk to me!)