

Ultraconvergence spaces exposed

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The aim of this note¹ is to expose a recent result [GMT26] at the intersection of

logic and **topology**.

In one sentence, the hope is to clarify how the model-theoretic construction of

ultraproducts

is entirely analogous to the topological concept of

ultraconvergence.

The main result is a *reconstruction* theorem for syntax out of semantics.

Note that I did not say the word ‘category’ anywhere in this description, and I will resist this urge for quite a few pages into this note. The reason for this methodology is that there already exist plenty of categorical expositions of these results [GMT26; Ham25; Saa25], and that I believe it may also be of interest to logicians or topologists who are less categorically minded.

As a *warm-up*, consider classical propositional logic on a countable set of variables:

- the formulas, regarded up to (any reasonable notion of) equivalence yield a *Boolean algebra* out of the syntax; let us call it A ;
- a *model* of this logic is given by a stream $x \in 2^\omega$, which uniquely yields an *evaluation homomorphism* $\llbracket - \rrbracket_x : A \rightarrow 2$;
- a *formula* of the logic denotes the set of evaluations making it true:

$$A \rightarrow \mathcal{P}(2^\omega)$$
$$\phi \mapsto \hat{\phi} := \{x \in 2^\omega \mid \llbracket \phi \rrbracket_x = 1\}.$$

¹This note is based on a lecture I gave at a Chocola meeting in Lyon on May 28, 2026. All of the work I report on here is joint with my former post-doc Jérémie Marquès and my PhD student Umberto Tarantino. For more details, proofs, and references, I refer to our paper [GMT26].

1. Question: Is this function $\widehat{(-)}$ injective?

Answer: Yes, provided the notion of ‘equivalence’ is complete for this semantics.

2. Question: What is the image of the function $\widehat{(-)}$?

Answer: The set of *clopen* subsets of 2^ω , in the product topology induced by the discrete topology on 2.

Let me emphasize that in the answer to the second question, a non-trivial *topology* appears. In light of the answer to the first question, this topology lets us *reconstruct* the algebra A .

The above warm-up is an instance of *Stone duality*. If you want to know more about this, there are books; my personal preference is [GG24] but I am biased.

My goal here is to tell a similar story for *first-order* logic, and to do so in a *not necessarily classical* setting.

In a paper titled *Stone duality for first-order logic*, M. Makkai [Mak87] identified that, in that setting, the role of ‘topology’ can be played by *ultraproducts*. Our main theorem is a variant of Makkai’s.² We will discuss its main ingredients, its statement, and some examples, here.

1 Ultraproducts

An *ultrafilter* is a complete theory of belonging.

Having said that, here is a more formal definition. An *ultrafilter on a set* I is a morphism $\mu : \mathcal{P}(I) \rightarrow 2$ of Boolean algebras. I will often identify an ultrafilter with $\mu^{-1}(1)$:

Notation. For $U \subseteq I$, ‘ $U \in \mu$ ’ means: $\mu(U) = 1$.

We write βI for the set of ultrafilters on I .

Example. For any element $i \in I$, the morphism η_i defined by

$$\eta_i(U) := \begin{cases} 1 & \text{if } i \in U \\ 0 & \text{if } i \notin U \end{cases}$$

is an ultrafilter. Such ultrafilters are called *principal*.

By (a weaker form of) the axiom of choice, any infinite set has a non-principal ultrafilter.

The more general notion of *ultrafilter in a Boolean algebra* will not directly concern us here, but let me note that, in the warm-up above, we saw that 2^ω is in bijection with the set of ultrafilters of the Boolean algebra A .

²It was obtained approximately at the same time by G. Saadia [Saa25] and A. Hamad [Ham25]. The three ArXiv publications took place within the scope of two months, with ours coming last, but containing a substantially different proof from the previous two. I regard this as circumstantial evidence of the naturality of the theorem, and the ripeness of the time for it.

Let $(M_i)_{i \in I}$ be an I -indexed family of sets, and let $\mu \in \beta I$. The *ultraproduct* of $(M_i)_{i \in I}$ along μ , denoted $\prod_{i \rightarrow \mu} M_i$, is a direct product of the M_i , but ‘up to’ the complete theory μ .

Formally, let $p : \sum_{i \in I} M_i \rightarrow I$ be the projection. An element of $\prod_{i \rightarrow \mu} M_i$ is an equivalence class of *partial sections* s of p such that $\text{dom}(s) \in \mu$, and

$$s \sim_\mu t \text{ if, and only if, } \{i \in \text{dom}(s) \cap \text{dom}(t) \mid s(i) = t(i)\} \in \mu .$$

Notation. An element of the ultraproduct is denoted $(s_i)_{i \rightarrow \mu}$ or, suggestively, $\lim_{i \rightarrow \mu} s_i$.

When the M_i are constantly equal to M , I will call an element of the ultraproduct $\prod_{i \rightarrow \mu} M$ a μ -family in M .

We have built a set out of an ultrafilter-indexed family of sets. Let me now extend this assignment to families of functions, yielding a *functor*

$$\prod_{i \rightarrow \mu} : \mathbf{Set}^I \rightarrow \mathbf{Set} .$$

Let $(f_i : M_i \rightarrow N_i)_{i \in I}$ be an I -indexed family of functions. The function

$$\begin{aligned} \prod_{i \rightarrow \mu} f_i : \prod_{i \rightarrow \mu} M_i &\rightarrow \prod_{i \rightarrow \mu} N_i \\ s &\mapsto \lambda i. f_i(s(i)) \end{aligned}$$

is well-defined on equivalence classes.

In logic, we can take the ultraproduct of *first-order structures* preserving the theory. More precisely:

Theorem 1 (Łoś).

$$\prod_{i \rightarrow \mu} M_i \models \phi \text{ if, and only if, } \{i \in I \mid M_i \models \phi\} \in \mu .$$

In this theorem, function symbols are interpreted in $\prod_{i \rightarrow \mu} M_i$ by the functoriality above.

We can view the interpretation of a unary predicate symbol in M_i as a function $P_i : M_i \rightarrow 2$, and functoriality then yields $\prod_{i \rightarrow \mu} P_i : \prod_{i \rightarrow \mu} M_i \rightarrow \prod_{i \rightarrow \mu} 2$. The set $\prod_{i \rightarrow \mu} 2$ contains two elements, so this is really a predicate on $\prod_{i \rightarrow \mu} M_i$. Explicitly, for an element $(x_i)_{i \rightarrow \mu}$ of $\prod_{i \rightarrow \mu} M_i$,

$$(x_i)_{i \rightarrow \mu} \text{ satisfies } \prod_{i \rightarrow \mu} P_i \text{ if, and only if, } \{i \in I \mid M_i \models P_i(x_i)\} \in \mu .$$

2 Ultraconvergence

Ultraconvergence is an ultimate way of tending towards.

Having said that, here is a more formal definition. Let μ be an ultrafilter on a set I , and let $(x_i)_{i \rightarrow \mu} \in \prod_{i \rightarrow \mu} X$ be a μ -family in X . For a point $x \in X$, we say that the family $(x_i)_{i \rightarrow \mu}$ *converges* to x with respect to μ if

$$\text{for any open } U \subseteq X, \text{ if } x \in U \text{ then } \{i \in I \mid x_i \in U\} \in \mu .$$

Notation. If $(x_i)_{i \in I}$ converges to x with respect to μ , we write

$$x \rightsquigarrow \lim_{i \rightarrow \mu} x_i$$

and, in the special case where $I = X$ and $x_i = i$ for each $i \in I$, we write

$$x \rightsquigarrow \lim \mu .$$

and we say that μ *converges to* x .

The direction of the arrow in this notation may be surprising. Here are two arguments in favor:

1. the notation evokes the direction of the implication that defines the concept ‘converges’;
2. the notation follows the usual direction of the *specialization preorder* of topological spaces, which can be defined in terms of the above notation, by: $x \preceq y$ if, and only if, $x \rightsquigarrow \lim_{* \rightarrow 1} y$. Here, 1 is the unique ultrafilter on the one-point set $\{*\}$.

A topological space is Hausdorff if, and only if, any ultrafilter on X converges to at most one point, and compact if, and only if, any ultrafilter on X converges to at least one point.

The set of ultrafilters βI carries a natural topology: For each $U \subseteq I$, define

$$\widehat{U} := \{\mu \in \beta I \mid U \in \mu\}$$

and take the collection of \widehat{U} as a basis.

Proposition 2. *The space βI is the free compact Hausdorff space on I , i.e., for any compact Hausdorff space X and any I -indexed family in X , $x = (x_i)_{i \in I} \in X^I$, there exists a unique continuous function $\tilde{x} : \beta I \rightarrow X$ making the following diagram commute:*

$$\begin{array}{ccc} \beta I & & \\ \uparrow \eta & \searrow \tilde{x} & \\ I & \xrightarrow{x} & X . \end{array}$$

The proposition can be used to turn β into a *monad* on \mathbf{Set} . The functorial action of β is given concretely, for $f : I \rightarrow J$ and $\mu \in \beta I$, by the formula

$$\beta f(\mu) = \{V \in \mathcal{P}(J) \mid f^{-1}V \in \mu\}.$$

The monad unit is η , and the monad multiplication is given by the fact that βI is itself a compact Hausdorff space.

If X is any compact Hausdorff space, then instantiating the proposition for the identity family $(x)_{x \in X} \in X^X$, we get a function $\beta X \rightarrow X$. This is an algebra for the monad β which completely encodes the topology on X :

Theorem 3 (Manes). *The category of compact Hausdorff spaces is equivalent to the category of algebras for the monad β .*

Makkai [Mak87] lifts Manes' theorem to the categorical level. Our work consists in lifting the following theorem to the categorical level:

Theorem 4 (Barr). *The category of topological spaces is equivalent to the category of relational modules for the monad β .*

It will be instructive to examine Barr's theorem in more detail.

If X is a topological space, then its associated relational β -module is the relation ' $x \rightsquigarrow \lim \mu$ ' between points of X and ultrafilters on X . This relation has the following three properties:

1. $x \rightsquigarrow \lim \eta_x$,
2. if $x \rightsquigarrow \lim_{i \rightarrow \mu} y_i$ and $y_i \rightsquigarrow \lim_{j \rightarrow v_i} z_{i,j}$, then $x \rightsquigarrow \lim_{(i,j) \in \sum_{i \rightarrow \mu} v_i} z_{i,j}$,
3. if $r : I \rightarrow J$ is a function,
 - (a) if $x \rightsquigarrow \lim_{j \rightarrow \beta r(\mu)} y_j$, then $x \rightsquigarrow \lim_{i \rightarrow \mu} y_{r(i)}$,
 - (b) if $x \rightsquigarrow \lim_{i \rightarrow \mu} y_{r(i)}$, then $x \rightsquigarrow \lim_{j \rightarrow \beta r(\mu)} y_j$.

I have a reason for not writing 'if and only if' in the third property! See below.

Let me first explain the new notation used in formulating these properties:

- for a μ -family $(y_i)_{i \rightarrow \mu}$ in X , we write $x \rightsquigarrow \lim_{i \rightarrow \mu} y_i$ as a notation for $x \rightsquigarrow \lim \beta y(\mu)$.
- if μ is an ultrafilter on I and, for each $i \in I$, v_i is an ultrafilter on J_i , then $\sum_{i \rightarrow \mu} v_i$ is the ultrafilter on $\sum_{i \in I} J_i$ defined, for $V \subseteq \sum_{i \in I} J_i$, by

$$\sum_{i \rightarrow \mu} v_i(V) := \mu(\{i \in I \mid v_i(V \cap J_i) = 1\}).$$

Note also that properties (1) and (2) generalize the fact that the specialization order \preceq of a topological space is reflexive and transitive, respectively.

An important part of Barr's theorem is that the topology on X can be recovered from the associated relational β -module: a subset $U \subseteq X$ is open if, and only if,

$$\text{for any } \mu\text{-family } (y_i)_{i \rightarrow \mu} \text{ in } X, \text{ if } x \rightsquigarrow \lim_{i \rightarrow \mu} y_i \text{ and } x \in U, \text{ then } \{i \in I \mid y_i \in U\} \in \mu.$$

The definition of *ultraconvergence space*³ in our paper [GMT26] lifts the three above properties of relational β -modules to a categorical level. An ultraconvergence space consists of:

- a class X of *points*;
- for any ultrafilter μ on a set I , any μ -family $(y_i)_{i \rightarrow \mu}$ in X , and any point x of X , a set of *ultra-arrows* from x to $(y_i)_{i \rightarrow \mu}$; we write $f : x \rightsquigarrow \lim_{i \rightarrow \mu} y_i$ to say that f is an ultra-arrow from x to $(y_i)_{i \rightarrow \mu}$;
 1. for any point x of X , an *identity* ultra-arrow $\text{id}_x : x \rightsquigarrow \lim_{i \rightarrow \mu} x$;
 2. for any ultra-arrows $f : x \rightsquigarrow \lim_{i \rightarrow \mu} y_i$ and $g_i : y_i \rightsquigarrow \lim_{j \rightarrow \nu_i} z_{i,j}$, a *composition*

$$(g_i)_{i \rightarrow \mu} \cdot f : x \rightsquigarrow \lim_{(i,j) \in \sum_{i \rightarrow \mu} \nu_i} z_{i,j}$$

3. for any function $r : I \rightarrow J$,
 - (a) for any ultra-arrow $f : x \rightsquigarrow \lim_{j \rightarrow \beta r(\mu)} y_j$, a *reindexing* ultra-arrow

$$r[f] : x \rightsquigarrow \lim_{i \rightarrow \mu} y_{r(i)};$$

subject to six axioms which I omit here. The important point is that this is (hopefully clearly) a lifting of the concept of relational β -module, except that (3b) is omitted. The latter point is explicated in a highly principled way in the recent follow-up paper by Umberto Tarantino together with a fellow PhD student Quentin Aristote [AT26].

A *continuous function* $F : X \rightarrow Y$ between ultraconvergence spaces assigns to each point $x \in X$ a point $f(x) \in Y$, and to each ultra-arrow $f : x \rightsquigarrow \lim_{i \rightarrow \mu} y_i$ in X an ultra-arrow $Ff : f(x) \rightsquigarrow \lim_{i \rightarrow \mu} f(y_i)$ in Y , in a way that preserves identity, composition, and renaming. The definition generalizes that of a *functor* between categories. In particular note that ‘continuous’ is additional structure on a function, and not just a property. We write $\mathbf{C}(X, Y)$ for the class of continuous functions $X \rightarrow Y$.

³[Saa25] calls the same concept ‘virtual ultracategory’, following a suggestion by Ivan Di Liberti, and [Ham25] calls a closely related concept ‘generalized ultracategory’.

3 Models

A logical theory can often be studied in terms of its models. Recalling the warm-up example, the models of a logical theory can be seen as a space. Let T be a (single-sorted) *geometric theory*, by which we mean a collection of implications of first-order formulas built from finite conjunction \wedge , infinite disjunction \bigvee , existential quantifier \exists , and equality $=$. A first-order structure is called a *model* of T if it verifies all of the implications in T . Due to the infinite disjunction, it may happen that T is consistent but does not have *any* models. However, we will always assume that T has enough models. We denote by $\text{Mod}(T)$ the collection of models of T .

The *ultraconvergence space* structure on $\text{Mod}(T)$ is defined by taking, for M a model of T and $(N_i)_{i \rightarrow \mu}$ a μ -family of models of T , the set of ultra-arrows from M to $(N_i)_{i \rightarrow \mu}$ to be the set of structure homomorphisms

$$M \rightarrow \prod_{i \rightarrow \mu} N_i .$$

The rest of the structure can be defined in a relatively straight-forward way. Note that, in the special case where T is the empty theory in the empty signature, this defines an ultraconvergence space structure on Set .

If ϕ is a formula in the language of T , say with k free variables, then its *semantics* is the assignment

$$\begin{aligned} \llbracket \phi \rrbracket &: \text{Mod}(T) \rightarrow \text{Set} \\ M &\mapsto \llbracket \phi \rrbracket_M := \{\bar{m} \in M^k \mid M \vDash \phi(\bar{m})\} . \end{aligned}$$

An important observation, analogous to Łoś' theorem, is that $\llbracket \phi \rrbracket$ is a *continuous* map of ultraconvergence spaces, in the sense that, for any ultra-arrow $M \rightarrow \prod_{i \rightarrow \mu} N_i$ of T -models, we get a naturally induced ultra-arrow $\llbracket \phi \rrbracket_M \rightarrow \prod_{i \rightarrow \mu} \llbracket \phi \rrbracket_{N_i}$. Our main theorem is that the converse is true:

Theorem 5 ([Ham25; Saa25; GMT26]). *Let T be a geometric theory with enough models. For any continuous $F : \text{Mod}(T) \rightarrow \text{Set}$, there exists a geometric formula ϕ such that $F \cong \llbracket \phi \rrbracket$.*

I will not define isomorphism of continuous maps, but it is what you would expect.

The assignments

$$T \mapsto \text{Mod}(T) \quad \text{and} \quad X \mapsto \mathbf{C}(X, \text{Set})$$

moreover have the property that

$$\{\text{continuous functions from } X \text{ to } \text{Mod}(T)\} \cong \{T\text{-models internal to } \mathbf{C}(X, \text{Set})\} .$$

To make precise what a ' T -model internal to' is, I will use some category theory.

4 Points

Let us first consider the propositional setting again, but this time for *geometric logic*. The corresponding algebraic structure is that of a *frame* (also known as *locale*): a complete lattice A in which $a \wedge \bigvee B = \bigvee_{b \in B} a \wedge b$ for all $a \in A$ and $B \subseteq A$.

A prototypical example of a frame is the collection of open sets $\mathcal{O}(X)$ of a topological space X . Any $x \in X$ induces a *neighborhood theory* $\eta_x : \mathcal{O}(X) \rightarrow 2$, defined by $\eta_x(U) := 1$ if, and only if $x \in U$. This is a homomorphism *of frames*, meaning that finite meets and arbitrary joins are preserved. Frame homomorphisms from a frame F to 2 are called *points* of a frame. They are analogous to ultrafilters of a Boolean algebra, but with an important difference: even assuming choice, a frame may not have any points whatsoever. This is closely related to the claim made above that a consistent geometric theory T may not have any models, as we will soon see.

For first-order geometric logic, the correct ‘algebraic structure’ is that of a (Grothendieck) *topos*. A topos is a category in which all constructs of geometric logic can be interpreted; more formally, it has finite limits (for interpreting \wedge), small colimits (for interpreting \bigvee), effective equivalence relations that are stable under pullback (for interpreting \exists and $=$), and a generating set (a set-theoretic size condition that I will ignore here).

Let \mathcal{E} be a topos. Any geometric formula ϕ admits an interpretation in \mathcal{E} , provided that the symbols of ϕ have been interpreted as objects of \mathcal{E} . This is analogous to the idea in our warm-up example that, once you fix a stream 2^ω interpreting the propositional variables, you also get an interpretation for arbitrary propositional formulas. We have replaced the Boolean algebra 2 by a topos \mathcal{E} . The usual semantics of first-order logic are obtained by taking the topos \mathcal{E} equal to **Set**.

Given a topos \mathcal{E} , we can now speak of *models* of a geometric theory T *internal to* \mathcal{E} : these are interpretations of the symbols occurring in T such that all sequents in the theory T become valid.

A functor $F : \mathcal{E} \rightarrow \mathcal{F}$ between toposes is called *the inverse image part of a geometric morphism* if it preserves finite limits and arbitrary colimits. Since that name is clearly too long and confusing, I will just call it a *homomorphism* here, by analogy with the case of frames.⁴ By analogy with the case of frames, a *point* of a topos \mathcal{E} is a homomorphism $\mathcal{E} \rightarrow \mathbf{Set}$.

A topos \mathcal{E}_T is said to *classify* a geometric theory T if, for any topos \mathcal{F} , the T -models internal to \mathcal{F} are the same as the homomorphisms $\mathcal{E}_T \rightarrow \mathcal{F}$. Any geometric theory has a classifying topos \mathcal{E}_T which is unique up to equivalence. In particular, the usual, set-based T -models are the same as the homomorphisms $\mathcal{E}_T \rightarrow \mathbf{Set}$. That is,

$$\mathrm{Mod}(T) \simeq \mathrm{Pt}(\mathcal{E}_T).$$

Any topos \mathcal{E} occurs as the classifying topos of some geometric theory. We thus always have, for any topos \mathcal{E} , an ultraconvergence structure on $\mathrm{Pt}(\mathcal{E})$, by transferring the one from $\mathrm{Mod}(T)$. More explicitly, when x is a point of \mathcal{E} and $(y_i)_{i \rightarrow \mu}$ is a μ -family of points of \mathcal{E} , an ultra-arrow $x \rightsquigarrow \lim_{i \rightarrow \mu} y_i$

⁴Topos theorists tend to let morphisms go in the other direction. What I do here is considering a topos as a *logos*, in the sense of [AJ21].

is, by definition, a natural transformation

$$x \Rightarrow \prod_{i \rightarrow \mu} y_i$$

where the ultraproduct on the right is computed point-wise. Note that $\prod_{i \rightarrow \mu} y_i$ is *not* necessarily a point of \mathcal{E} , but it is always a coherent functor from \mathcal{E} to \mathbf{Set} .

We can now give an equivalent topos-theoretic perspective on the main theorem:

Theorem 6. *Let \mathcal{E} be a topos with enough points. For any continuous $F : \mathbf{Pt}(\mathcal{E}) \rightarrow \mathbf{Set}$, there exists an object ϕ in \mathcal{E} such that $F \cong \llbracket \phi \rrbracket$.*

Here, for an object ϕ in \mathcal{E} , the associated continuous map is just

$$\begin{aligned} \llbracket \phi \rrbracket : \mathbf{Pt}(\mathcal{E}) &\rightarrow \mathbf{Set} \\ x &\mapsto x\phi \\ x \rightsquigarrow \lim_{i \rightarrow \mu} y_i &\mapsto x\phi \rightarrow \prod_{i \rightarrow \mu} y_i\phi. \end{aligned}$$

Just as for models, we also get a connection between toposes and ultraconvergence spaces:

$$\mathbf{Pt} : \mathbf{Topos} \rightleftarrows \mathbf{UCSp}_b : \mathbf{C}(-, \mathbf{Set}).$$

The subscript b is due to the set-theoretic size condition on toposes already alluded to above. This is a pseudoadjunction, in the sense that

$$\{\text{continuous functions } X \rightarrow \mathbf{Pt}(\mathcal{E})\} \cong \{\text{homomorphisms } \mathcal{E} \rightarrow \mathbf{C}(X, \mathbf{Set})\}.$$

Note that the pseudoadjunction looks contravariant but is in fact covariant, since by definition geometric morphisms in the category of toposes are defined as going in the reverse direction of homomorphisms.

Our main theorem implies that the counit of this adjunction $\llbracket - \rrbracket : \mathcal{E} \rightarrow \mathbf{C}(\mathbf{Pt}(\mathcal{E}), \mathbf{Set})$, is an equivalence. Therefore, in the pseudoadjunction, the left adjoint \mathbf{Pt} is pseudo-fully-faithful. In other words, a topos with enough points can always be reconstructed from its ultraconvergence space as the topos of continuous functions from its space of points to \mathbf{Set} .

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