Stone duality for Boolean algebras

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The aim of this note is to give a detailed proof of Stone duality for Boolean algebras [1] to facilitate its formalization in Mathlib. Note that Stone actually proved something more general for bounded distributive lattices [2]. (Below, I will cut everything into very small numbered lemmas and definitions which is not how I usually like to write math, but it seemed like it might be useful for organizing the formalization. TODO: put this in the lean blueprint format.)

1 Definitions and statement

I first recall some definitions from Mathlib. A subset S of a topological space is *preconnected* if, whenever U and V are open subsets such that $S \subseteq U \cup V$, $S \cap U \neq \emptyset$ and $S \cap V \neq \emptyset$, we have $S \cap U \cap V \neq \emptyset$. A topological space is *totally disconnected* if any preconnected subset is a subsingleton. A *Boolean space* is a compact Hausdorff totally disconnected space.[1] The category **BoolSp** of Boolean spaces is the full subcategory of topological spaces based on Boolean spaces.

A *Boolean algebra* is by definition a bounded distributive lattice with complements. This means that it is a structure of the form $(A, \vee, \wedge, 0, 1, \neg)$ satisfying a number of axioms. We do not really care what the axioms are precisely. What is important to know is that, for any set X, the structure $(\mathcal{P}(X), \cup, \cap, \emptyset, X, (-)^{\mathsf{c}})$ is a Boolean algebra. In particular, I write $\mathbf{2} = \{0, 1\}$ for the unique two-element Boolean algebra; it can be seen that it is a Boolean algebra because it is the power set algebra of a singleton set. A Boolean algebra *homomorphism* is a structure-preserving function between Boolean algebras. In fact, to be a homomorphism, it suffices to preserve $\vee$, 0, and $\neg$, because the operations 1 and $\wedge$ are term-definable from those. In particular, for a subset S of a Boolean algebra A to be a subalgebra, it suffices for S to contain 0, and be closed under the operations $\vee$ and $\neg$. An *ideal* of a Boolean algebra is a subset I which contains 0 and is such that $a \vee b \in I$ if, and only if, both $a \in I$ and $b \in I$ (this is equivalent to the usual definition for rings but formulated in a more convenient way for Boolean algebras). An ideal is *proper* if it does not contain 1. The *partial order* on a Boolean algebra A can be defined by $a \leq b$ iff $a \vee b = b$, or equivalently iff $a \wedge b = a$. Our aim is to prove:

Theorem 1.1. *The categories* **BoolAlg** *and* **BoolSp** *are dually equivalent.*

It will also be useful and interesting to actually have concrete definitions of the dual equivalence functors both ways, as we will do below.

[1] What I call Boolean space is called `Profinite` in Mathlib and is sometimes called 'Stone space' by other mathematicians.

Remark 1.2. An alternative proof, which is shorter on paper but does not give explicit access to the definition of the dual equivalence functors, would be to use the fact that any fully faithful essentially surjective functor is part of an equivalence (this is in Mathlib). However, the inverse part of the equivalence that is produced by that fact only exists thanks to an application of axiom of choice, and it seems like it would be hard to work with. I could of course be wrong about this, and implementing this road towards the proof might be an interesting alternative experiment. It may have the advantage of avoiding getting into the weeds of natural transformation arguments as I need to do in Propositions 4.6 to 4.8 below.

2 Two useful facts

We will need the following fundamental fact about Boolean algebras which is sometimes called the ultrafilter principle. It could also be deduced from the ring-theoretic fact (probably in Mathlib) that any non-unit element of a ring is in some maximal ideal, but a direct proof is not hard.

Lemma 2.1. *Let A be a Boolean algebra. For any $a \in A \setminus \{1\}$, there exists a homomorphism $x \colon A \to \mathbf{2}$ such that $x(a) = 0$.*

Proof. By Zorn's lemma, let I be a maximal element of the set of proper ideals of A which contain a. Define $x(b) = 0$ iff $b \in I$. Clearly, $x(a) = 0$; we need to check that x is a homomorphism. The equalities $x(0) = 0$ and $x(b \vee b') = x(b) \vee x(b')$ are easy to check from the defining properties of an ideal. To see that $x(\neg b) = \neg x(b)$ for any $b \in A$, the crucial observation is that if $b \notin I$ and $\neg b \notin I$, then it is possible to enlarge I by adding b to it and generating an ideal I', and I' will still be proper because $\neg b \notin I$. By maximality of I this is impossible. We thus get that, for any $b \in A$, one of b and $\neg b$ must be in I, and they can never be both in I, since that would give $1 = b \vee \neg b$ in I, again contradicting that I is proper. It now follows from the definitions that $x(\neg b) = \neg x(b)$. □

The following equivalent definition of Boolean spaces is convenient.

Lemma 2.2. *A compact Hausdorff space X is totally disconnected if, and only if, for any distinct $x, y \in X$, there exists a clopen set $K \subseteq X$ such that $x \in K$ and $y \notin K$.*

Proof. The sufficiency is in Mathlib already. It is also in Mathlib that a compact Hausdorff totally disconnected space is totally separated. (Recall that a space X is called *totally separated* if, for any distinct $x, y \in X$, there exist disjoint open sets $U, V \subseteq X$ such that $x \in U$ and $y \in V$.) Finally, in any totally separated space, one may actually separate distinct points by clopens, and this is also in Mathlib. □

3 The functors

We define the two functors that make up Stone duality for Boolean algebras: $\mathsf{Clp} \colon \mathbf{BoolSp} \to \mathbf{BoolAlg}$ and $\mathsf{Spec} \colon \mathbf{BoolAlg} \to \mathbf{BoolSp}$.

From spaces to algebras. If X is a Boolean space, write $\mathsf{Clp}(X)$ for the set of clopen subsets of X.

Proposition 3.1. $\mathsf{Clp}(X)$ *is a Boolean subalgebra of the Boolean algebra* $\mathcal{P}(X)$.

Proof. Finite unions and complements of clopen sets are clopen, and the empty set is clopen. This is in Mathlib. ☐

Proposition 3.2. *If* $f : X \to Y$ *is a continuous function between Boolean spaces, then* $f^{-1} : \mathsf{Clp}(Y) \to \mathsf{Clp}(X)$ *is a Boolean algebra homomorphism.*

Proof. $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(K^{\mathsf{c}}) = f^{-1}(K)^{\mathsf{c}}$, and $f^{-1}(K_1 \cup K_2) = f^{-1}(K_1) \cup f^{-1}(K_2)$ for any clopens K, K_1, K_2 of Y. ☐

One sometimes writes $\mathsf{Clp}(f)$ for f^{-1} but the notation is a bit heavy.

Proposition 3.3. *The assignments* $X \mapsto \mathsf{Clp}(X)$ *and* $f \mapsto f^{-1}$ *give a contravariant functor* **BoolSp** $\to$ **BoolAlg**.

Proof. The assignments are well-defined by Proposition 3.1 and Proposition 3.2. It is a contravariant functor because $\mathsf{id}_X^{-1}(K) = K$ and $(f \circ g)^{-1}(K) = g^{-1}(f^{-1}(K))$ for any clopen K. ☐

From algebras to spaces. If A is a Boolean algebra, write $\mathsf{Spec}(A)$ for the set of homomorphisms from A to $\mathbf{2}$. We equip $\mathsf{Spec}(A)$ with the subspace topology induced by the $|A|$-fold power $\mathbf{2}^A$, where $\mathbf{2}$ has the discrete topology. Define, for any $a \in A$,

$$U_a \stackrel{\text{def}}{=} \{x \in \mathsf{Spec}(A) \mid x(a) = 1\} .$$

Lemma 3.4. *For any* $a \in A$*, the complement of* U_a *is equal to* $U_{\neg a}$.

Proof. Note that $x \in (U_a)^{\mathsf{c}} \iff x(a) \neq 1 \iff x(a) = 0 \iff x(\neg a) = 1 \iff x \in U_{\neg a}$. ☐

Lemma 3.5. *For any* $a, b \in A$ *we have* $U_a \cup U_b = U_{a \vee b}$*, and moreover* $U_0 = \emptyset$.

Proof. For any $x \in \mathsf{Spec}(A)$, we have

$$x(a \vee b) = 1 \iff x(a) \vee x(b) = 1 \iff x(a) = 1 \text{ or } x(b) = 1 .$$

This shows that $U_a \cup U_b = U_{a \vee b}$. To see that $U_0 = \emptyset$ just notice that $x(0) = 0 \neq 1$. ☐

Lemma 3.6. *For any* $a \in A$*, the set* U_a *is clopen in* $\mathsf{Spec}(A)$.

Proof. For any $a \in A$, the set U_a is open because it is $\pi_a^{-1}(1)$ where π_a denotes the restriction of the projection function from $\mathsf{Spec}(A)$ to $\mathbf{2}$ on the coordinate a, and this projection function is continuous by the definitions of product and subspace topology. The set U_a is closed because its complement is equal to $U_{\neg a}$ by Lemma 3.4, and we just showed that any U_b is open. ☐

Proposition 3.7. *The topology on* $\mathsf{Spec}(A)$ *is generated by the basis* $(U_a)_{a \in A}$ *of clopen sets.*

Proof. In general, the topology on a product $\prod_{i \in I} X_i$ is generated by the sets $\pi_i^{-1}(U)$ where $i \in I$ and $U \subseteq X_i$ ranges over a basis for the open sets of X_i. Now in the case of $\mathbf{2}^A$, this implies that the topology has as a basis the sets $\pi_a^{-1}(0)$ and $\pi_a^{-1}(1)$, as a ranges over the elements of A. But Lemma 3.4 gives that $\pi_a^{-1}(0) = \pi_{\neg a}^{-1}(1)$, so it actually suffices to take the sets $U_a = \pi_a^{-1}(1)$ as a ranges over the elements of A. ☐

Theorem 3.8. *The topological space* $\mathsf{Spec}(A)$ *is a Boolean space.*

Proof. If $x, y \in \mathsf{Spec}(A)$ are distinct, then there is an element $a \in A$ such that $x(a) \neq y(a)$. Without loss, say that $x(a) = 1$ and $y(a) = 0$. Then $x \in U_a$ and $y \notin U_a$. Thus, any two distinct elements are separated by a clopen set. This suffices to conclude that $\mathsf{Spec}(A)$ is totally disconnected by a result that is in Mathlib already, and it also clearly implies that the space is Hausdorff. It remains to prove that $\mathsf{Spec}(A)$ is compact. This is where a weak form of axiom of choice must be used. We show that the set of homomorphisms is closed as a subspace of $\mathbf{2}^{|A|}$, and it is therefore compact, since $\mathbf{2}^{|A|}$ is compact by Tychonoff's Theorem. In order to see that $\mathsf{Spec}(A)$ is closed in $\mathbf{2}^{|A|}$, notice that

$$\mathsf{Spec}(A) = \bigcap_{a,b \in A} J_{a,b} \cap \bigcap_{a \in A} N_a \cap Z \ ,$$

where $J_{a,b} \stackrel{\text{def}}{=} \{x \in \mathbf{2}^A \ : \ x(a \vee b) = x(a) \vee x(b)\}$, $N_a \stackrel{\text{def}}{=} \{x \in \mathbf{2}^A \ : \ x(\neg a) = \neg x(a)\}$, $Z \stackrel{\text{def}}{=} \{x \in \mathbf{2}^A \ : \ x(0) = 0\}$. Each of these sets is clopen because its definition only depends on a finite number of coordinates of x. For example, to spell this out a bit more, for any $a, b \in A$,

$$J_{a,b} = (\pi_{a \vee b}^{-1}(0) \cap \pi_a^{-1}(0) \cap \pi_b^{-1}(0)) \cup (\pi_{a \vee b}^{-1}(1) \cap (\pi_a^{-1}(1) \cup \pi_b^{-1}(1))) \ ,$$

where $\pi_c \colon \mathbf{2}^A \to \mathbf{2}$ is the projection onto the c coordinate. □

Definition 3.9. Given a homomorphism $f \colon A \to B$ we define $f_* \colon \mathsf{Spec}(B) \to \mathsf{Spec}(A)$ as the map that sends $y \colon B \to \mathbf{2}$ to $y \circ f \colon A \to \mathbf{2}$.

This is well-defined: Since $y \circ f$ is the composition of two homomorphisms, it is again a homomorphism, showing that f_* is a well-defined function.

Proposition 3.10. *For any Boolean algebra homomorphism* $f \colon A \to B$, $f_* \colon \mathsf{Spec}(B) \to \mathsf{Spec}(A)$ *is a continuous function.*

Proof. For any $a \in A$ and $y \in \mathsf{Spec}(B)$, we have

$$y \in f_*^{-1}(U_a) \iff f_*(y) \in U_a \iff y(f(a)) = 1 \iff y \in U_{f(a)} \ ,$$

so that the inverse image $f_*^{-1}(U_a)$ of any basic open set is again a (basic) open. □

Proposition 3.11. *The assignment* $f \mapsto f_*$ *is contravariant functorial from* **BoolAlg** *to* **BoolSp**.

Proof. If $f \colon A \to B$ and $g \colon B \to C$ then for any $y \in \mathsf{Spec}(C)$,

$$(\mathsf{id}_C)_*(y) = y \circ \mathsf{id}_C = y, \quad (g \circ f)_*(y) = y \circ g \circ f = f_*(g_*(y)) \ .$$

□

One sometimes writes $\mathsf{Spec}\, f$ for f_* but the notation is a bit heavy.

4 The equivalence

Definition 4.1. Let A be a Boolean algebra. We write η_A for the Boolean homomorphism that sends $a \in A$ to $U_a \in \mathsf{Clp}(\mathsf{Spec}(A))$.

It is indeed a homomorphism by Lemma 3.4 and Lemma 3.5.

Proposition 4.2. *The function η_A is bijective, and thus an isomorphism in* **BoolAlg**.

Proof. The function η_A is injective: Lemma 2.1 gives that if $U_a = \mathsf{Spec}(A)$ then $a = 1$. Now if $a \neq b$ then $(\neg a \vee b) \wedge (\neg b \vee a) \neq 1$, from which one gets that $\eta(a) \neq \eta(b)$ (this is the Boolean algebra version of the usual argument that a ring homomorphism is injective if it has zero kernel).

The function η_A is surjective: Let $K \subseteq \mathsf{Spec}(A)$ be clopen. By Proposition 3.7, since K is open, there exists $I \subseteq A$ such that $K = \bigcup_{a \in I} U_a$. Since K is closed and $\mathsf{Spec}(A)$ is compact, K is compact. Pick a finite subset $F \subseteq I$ such that $K = \bigcup_{a \in F} U_a$. By Lemma 3.5, we get $K = U_{a_0} = \eta_A(a_0)$, where $a_0 \stackrel{\text{def}}{=} \bigvee F$. □

Definition 4.3. Let X be a Boolean space. We write ϵ_X for the function $X \to \mathsf{Spec}(\mathsf{Clp}(X))$ that sends $x \in X$ to the homomorphism $\epsilon_X(x) \colon \mathsf{Clp}(X) \to \mathbf{2}$ defined by sending any $K \in \mathsf{Clp}(X)$ to 1 if $x \in K$, and to 0 otherwise.

Lemma 4.4. *Let X be a Boolean space. For any $K \in \mathsf{Clp}(X)$, we have $\epsilon_X^{-1}(U_K) = K$.*

Proof. Note that, for any $x \in X$,

$$x \in \epsilon_X^{-1}(U_K) \iff \epsilon_X(x) \in U_K \iff \epsilon_X(x)(K) = 1 \iff x \in K\ .$$ □

Proposition 4.5. *The function ϵ_X is a homeomorphism, for any Boolean space X.*

Proof. ϵ_X is continuous: By Proposition 3.7 and a general fact about continuous functions, it suffices to check that $\epsilon_X^{-1}(U_K)$ is open for every $K \in \mathsf{Clp}(X)$. This set is equal to K by Lemma 4.4, and thus open.

ϵ_X is injective: For any $x, y \in X$, if $x \neq y$, then there is $K \in \mathsf{Clp}(X)$ such that $x \in K$ and $y \notin K$, using the characterization of Boolean spaces given in Lemma 2.2. Thus, $\epsilon_X(x)(K) = 1$ while $\epsilon_X(x)(K) = 0$.

ϵ_X is surjective: Let $\xi \in \mathsf{Spec}(\mathsf{Clp}(X))$. Consider $F \stackrel{\text{def}}{=} \xi^{-1}(1) = \{K \in \mathsf{Clp}(X) \mid \xi(K) = 1\}$. Note that, for any finite collection $\{K_1, \dots, K_n\} \subseteq F$, we also have $\bigcap_{i=1}^n K_i \in F$, and in particular the intersection is non-empty, since $\emptyset \notin F$. Thus, since X is compact and F is a filter of closed(-and-open) sets which does not contain $\emptyset$, the intersection $\bigcap F$ is non-empty. Pick $x \in \bigcap F$. We claim that $\epsilon_X(x) = \xi$. Let $K \in \mathsf{Clp}(X)$ be arbitrary, we need to show that $\epsilon_X(x)(K) = \xi(K)$. There are two cases. If $K \in F$, then $x \in K$, so $\epsilon_X(x)(K) = 1 = \xi(K)$. Otherwise, $K \notin F$. Then $\xi(K) = 0$, and, using that ξ is a homomorphism, we have $\xi(K^{\mathsf{c}}) = \neg\xi(K) = 1$. Thus, $K^{\mathsf{c}} \in F$, from which we get that $x \in K^{\mathsf{c}}$, so $\epsilon_X(x)(K) = 0$, as required.

Now ϵ_X is a homeomorphism because it is a continuous bijection from a compact to a Hausdorff space. □

Proposition 4.6. *The assignment $A \mapsto \eta_A$ is a natural transformation from the identity functor on* **BoolAlg** *to the functor $\mathsf{Clp} \circ \mathsf{Spec}$.*

Proof. Let $h \colon A \to B$ be a homomorphism and $a \in A$. For any $x \in \mathsf{Spec}(B)$, we have

$$x \in \eta_B(h(a)) \iff x(h(a)) = 1 \iff x \circ h \in \eta_A(a) \iff x \in (\mathsf{Clp} \circ \mathsf{Spec})(h)(\eta_A(a))\ ,$$

where the last equivalence follows from the definitions of Clp and Spec on morphisms. We conclude that $\eta_B(h(a)) = (\mathsf{Clp} \circ \mathsf{Spec})(h)(\eta_A(a))$, as required. ☐

Proposition 4.7. *The assignment $X \mapsto \epsilon_X$ is a natural transformation from the identity functor on* **BoolSp** *to the functor* $\mathsf{Spec} \circ \mathsf{Clp}$.

Proof. Let $f : X \to Y$ be a continuous function and $x \in X$. Write $\xi_1 \stackrel{\text{def}}{=} \epsilon_Y(f(x))$ and $\xi_2 \stackrel{\text{def}}{=} ((\mathsf{Spec} \circ \mathsf{Clp})(f))(\epsilon_X(x))$. For any $K \in \mathsf{Clp}(Y)$, we have

$$\xi_1(K) = 1 \iff f(x) \in K \iff x \in f^{-1}(K) \iff \xi_2(K) = 1 \ ,$$

where the last equivalence again follows from the definitions of Spec and Clp on morphisms. Thus, $\xi_1 = \xi_2$. ☐

According to the definition of categorical equivalence in Mathlib, we finally need to prove one of the triangle laws for the natural isomorphisms η and ϵ, namely the one that says that the composite natural transformation $\mathsf{Spec} \Rightarrow \mathsf{Spec}\,\mathsf{Clp}\,\mathsf{Spec} \Rightarrow \mathsf{Spec}$ is the identity.

Proposition 4.8. *The triangle law for η and ϵ holds.*

Proof. Let A be a Boolean algebra. We write $f \stackrel{\text{def}}{=} \mathsf{Spec}\,\eta_A : \mathsf{Spec}\,\mathsf{Clp}\,\mathsf{Spec}\,A \to \mathsf{Spec}\,A$ and $g \stackrel{\text{def}}{=} \epsilon_{\mathsf{Spec}\,A} : \mathsf{Spec}\,A \to \mathsf{Spec}\,\mathsf{Clp}\,\mathsf{Spec}\,A$; we need to prove that $f \circ g$ is the identity.[2] Let $x \in \mathsf{Spec}\,A$. We want to show that $x = (f \circ g)(x)$. Write $y \stackrel{\text{def}}{=} g(x)$. By definition of g, we have, for any $K \in \mathsf{Clp}\,\mathsf{Spec}\,A$, that $y(K) = 1 \iff x \in K$. By definition of f, we have, for any $a \in A$, that $f(y)(a) = 1 \iff y(\eta_A(a)) = 1$. Combining these two facts, we see that, for any $a \in A$, $f(y)(a) = 1 \iff x \in \eta_A(a) \iff x(a) = 1$, by definition of η_A. Thus, $f(g(x)) = f(y) = x$. ☐

[2]Note that I had to reverse direction of morphisms with respect to what is written in the mathlib documentation, because Spec and Clp are contravariant functors.