

A proof of Schützenberger’s theorem

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1 Introduction

The set of *starfree* languages over a finite alphabet Σ is inductively defined as the smallest set of languages such that

- the languages \emptyset and $\{\epsilon\}$ are starfree;
- for any letter $a \in \Sigma$, the language $\{a\}$ is starfree;
- for any two starfree languages K and L , the languages $K \cup L$, $\Sigma^* \setminus L$, and $K \cdot L$ are starfree.

Recall that a finite monoid M is *aperiodic* if, and only if, there exists $\ell \in \mathbb{N}$ such that, for every $m \in M$, $m^\ell = m^{\ell+1}$. Many different, equivalent formulations of the definition exist, see e.g., [1, Lem. 1.144], but we do not need them.

Also recall that the *syntactic congruence* of a language $L \subseteq \Sigma^*$ is the equivalence relation \sim_L on Σ^* defined, for $u, v \in \Sigma^*$, by

$$u \sim_L v \iff \text{for any } \alpha, \beta \in \Sigma^*, \alpha u \beta \in L \text{ if, and only if, } \alpha v \beta \in L .$$

The *syntactic monoid* of L is $M_L := \Sigma^* / \sim_L$. The monoid M_L is finite if, and only if, L is regular.

Theorem 1 (Schützenberger, 1965). *Let L be a language. Then L is starfree if, and only if, the syntactic monoid of L is finite and aperiodic.*

The theorem has a rich history and many different proofs, see [4] for an account. Our aim here is to give a short and self-contained proof. Nothing here is original, and our account is closest to the ones in [1, Sec. 1.12] and [3, Ch. VI].

2 Green’s relations in a finite aperiodic monoid

We introduce a few important definitions and facts about them.

Definition 2. Let M be a monoid and let $x, y \in M$. The element x is a *factor* of y if $y \in MxM$, i.e., if there exist $\alpha, \beta \in M$ such that $y = \alpha x \beta$. In this case we write $y \leq_{\mathcal{J}} x$.

The element x is a *prefix* of y if $y \in xM$, i.e., if there exists $\beta \in M$ such that $y = x\beta$. In this case we write $y \leq_{\mathcal{R}} x$.

The element x is a *suffix* of y if $y \in Mx$, i.e., if there exists $\alpha \in M$ such that $y = \alpha x$. In this case we write $y \leq_{\mathcal{L}} x$.

The elements x and y are *factor-equivalent* if both $y \leq_{\mathcal{J}} x$ and $x \leq_{\mathcal{J}} y$; in this case we write $x\mathcal{J}y$. The *prefix-equivalence* (\mathcal{R}) and *suffix-equivalence* (\mathcal{L}) relations are defined analogously.

As a useful notation, we write $x <_{\mathcal{J}} y$ to mean $x \leq_{\mathcal{J}} y$ and $y \not\leq_{\mathcal{J}} x$.

Exercise 3. The relations $\leq_{\mathcal{L}}$, $\leq_{\mathcal{R}}$ and $\leq_{\mathcal{J}}$ are preorders, i.e., reflexive and transitive.

In the free monoid Σ^* , the equivalence relations \mathcal{L} , \mathcal{R} , and \mathcal{J} are all trivial, but in other monoids they are usually not.

Exercise 4. Work out in detail what the relations $\leq_{\mathcal{R}}$, $\leq_{\mathcal{L}}$ and $\leq_{\mathcal{J}}$ look like for the syntactic monoid of the language $(ab)^*$.

Exercise 5. Prove that $\leq_{\mathcal{L}}$ is right-invariant, i.e., for any $x, y, m \in M$, if $y \leq_{\mathcal{L}} x$, then $ym \leq_{\mathcal{L}} xm$. Similarly, prove that $\leq_{\mathcal{R}}$ is left-invariant. Is $\leq_{\mathcal{L}}$ also left-invariant? Is $\leq_{\mathcal{J}}$ invariant on either side? Prove or give counterexamples.

Lemma 6 (Preservation). *Let $\phi: M \rightarrow N$ be a homomorphism between monoids, and let $m, m' \in M$. For each $\mathcal{X} \in \{\mathcal{R}, \mathcal{L}, \mathcal{J}\}$, if $m \leq_{\mathcal{X}} m'$, then $\phi(m) \leq_{\mathcal{X}} \phi(m')$.*

Proof. We show the case $\mathcal{X} = \mathcal{J}$, the other two cases are proved analogously. Suppose $m \leq_{\mathcal{J}} m'$. Pick $\alpha, \beta \in M$ such that $m = \alpha m' \beta$. Then, using that ϕ is a homomorphism, we have $\phi(m) = \phi(\alpha)\phi(m')\phi(\beta)$, so that $\phi(m) \leq_{\mathcal{J}} \phi(m')$. \square

Lemma 7 (Simplification). *Let M be a finite aperiodic monoid and let $x, \alpha, \beta \in M$. If $x = \alpha x \beta$, then $x = \alpha x$ and $x = x \beta$.*

Proof. Pick $\ell \in \mathbb{N}$ such that, for every $m \in M$, $m^\ell = m^{\ell+1}$. By iteratively applying the equality $x = \alpha x \beta$ for ℓ iterations, we have $x = \alpha^\ell x \beta^\ell$. Using this equation twice, and the equation $\beta^\ell = \beta^\ell \beta$, we get

$$x = \alpha^\ell x \beta^\ell = \alpha^\ell x \beta^\ell \beta = x \beta .$$

The proof that $x = \alpha x$ is analogous. \square

Lemma 8 (Stability). *Let M be a finite aperiodic monoid. For any $x, y, \alpha, \beta \in M$, if $x = \alpha y \beta$, then*

1. *if $y \leq_{\mathcal{L}} x$, then $x = \alpha y$;*
2. *if $y \leq_{\mathcal{R}} x$, then $x = y \beta$;*
3. *if both $y \leq_{\mathcal{L}} x$ and $y \leq_{\mathcal{R}} x$, then $x = y$.*

Proof. (1) Pick $\gamma \in M$ such that $y = \gamma x$. Then $x = \alpha y \beta = \alpha \gamma x \beta$. By the Simplification Lemma, $x = \alpha \gamma x = \alpha y$.

(2) Symmetric.

(3) By the first hypothesis and item 1, we have $x = \alpha y$. Now by the second hypothesis, $y \leq_{\mathcal{R}} x = \alpha y$, so applying item 2 to $x := \alpha y$ and $\beta := 1$, we get $x = y \cdot 1 = y$. \square

Corollary 9 below reformulates Lemma 8 in a more usable form.

Corollary 9. *Let M be a finite aperiodic monoid and let $x, y \in M$ be such that $x \leq_{\mathcal{J}} y$. Then:*

1. *if $y \leq_{\mathcal{L}} x$, then $x \leq_{\mathcal{L}} y$;*
2. *if $y \leq_{\mathcal{R}} x$, then $x \leq_{\mathcal{R}} y$;*
3. *(Strong \mathcal{H} -triviality) if both $y \leq_{\mathcal{L}} x$ and $y \leq_{\mathcal{R}} x$, then $x = y$.*

Proof. The main hypothesis implies that we can pick $\alpha, \beta \in M$ as in the hypothesis of Lemma 8. \square

We explain the name of Corollary 9.3. In the literature on finite monoids, the Green relation $\leq_{\mathcal{R}} \cap \leq_{\mathcal{L}}$ is denoted $\leq_{\mathcal{H}}$, and \mathcal{H} is the associated equivalence relation. Since $\leq_{\mathcal{H}} \subseteq \leq_{\mathcal{J}}$, Corollary 9.3 implies in particular that, in a finite aperiodic monoid, the equivalence classes of \mathcal{H} are trivial.

Lemma 10 (Unit). *Let M be a finite aperiodic monoid, and $m \in M$. Then:*

1. *if $1 \leq_{\mathcal{L}} m$, then $m = 1$;*
2. *if $1 \leq_{\mathcal{R}} m$, then $m = 1$;*
3. *if $1 \leq_{\mathcal{J}} m$, then $m = 1$.*

Proof. (1) Suppose $1 \leq_{\mathcal{L}} m$. Since $m = 1 \cdot 1 \cdot m$, Lemma 8.1 gives $m = 1 \cdot 1 = 1$.

(2) Symmetric.

(3) Suppose $1 \leq_{\mathcal{J}} m$. Pick $\alpha, \beta \in M$ such that $1 = \alpha m \beta$. Then $1 \leq_{\mathcal{R}} \alpha m$, so by (2), $\alpha m = 1$. Thus, $1 \leq_{\mathcal{L}} m$, and applying (1), we have $m = 1$. \square

3 From aperiodic to starfree

In this section, we will prove the following.

Theorem 11. *Let M be a finite aperiodic monoid and let $\phi: \Sigma^* \rightarrow M$ be a homomorphism. For any subset T of M , the language $\phi^{-1}(T)$ is starfree.*

The right-to-left direction of Theorem 1 follows immediately from Theorem 11 by taking M to be the syntactic monoid of L , and T the image of L under the morphism $\Sigma^* \rightarrow M_L$.

For the rest of this section, fix a finite aperiodic monoid $(M, \cdot, 1)$, and a homomorphism $\phi: \Sigma^* \rightarrow M$. For any $m \in M$, write $L_m := \phi^{-1}(m)$. We aim to show that L_m is starfree for every $m \in M$. We first treat the case $m = 1$ separately.

Proposition 12. *Define $\Sigma_{\neq 1} := \{\mathbf{a} \in \Sigma \mid \phi(\mathbf{a}) \neq 1\}$. Then*

$$L_1 = \Sigma^* \setminus \Sigma^* \Sigma_{\neq 1} \Sigma^* ,$$

so, in particular, L_1 is starfree.

Proof. We prove the two inclusions. Let $w \in \Sigma^*$.

For the left-to-right direction, suppose that $\phi(w) = 1$, and let $\mathbf{a} \in \Sigma$ be any letter that appears in w . Then $1 = \phi(w) \leq_{\mathcal{J}} \phi(\mathbf{a})$ by the Preservation Lemma. Therefore, by Lemma 10.3, we get $\phi(\mathbf{a}) = 1$, so $\mathbf{a} \notin \Sigma_{\neq 1}$.

For the right-to-left direction, if $w \in \Sigma^* \setminus \Sigma^* \Sigma_{\neq 1} \Sigma^*$, then $\phi(\mathbf{a}) = 1$ for all letters \mathbf{a} that appear in w , and thus $\phi(w) = 1$. \square

Convention. For the rest of this section, letters of the alphabet are written in boldface font, while their respective images under ϕ are written in italic font. That is, if $\Sigma = \{\mathbf{a}, \mathbf{b}, \dots\}$, then $a := \phi(\mathbf{a})$, $b := \phi(\mathbf{b})$, et cetera.

Definition 13. Let $m \in M \setminus \{1\}$. We define four sets

$$\begin{aligned} P_m &:= \{(n, \mathbf{a}) \in M \times \Sigma \mid n\mathbf{a}\mathcal{R}m \text{ and } n \not\leq_{\mathcal{R}} m\}, \\ S_m &:= \{(\mathbf{a}, n) \in \Sigma \times M \mid \mathbf{a}n\mathcal{L}m \text{ and } n \not\leq_{\mathcal{L}} m\}, \\ F_m &:= \{(\mathbf{a}, n, \mathbf{b}) \in \Sigma \times M \times \Sigma \mid m \leq_{\mathcal{J}} \mathbf{a}n, m \leq_{\mathcal{J}} n\mathbf{b}, \text{ and } m \not\leq_{\mathcal{J}} \mathbf{a}n\mathbf{b}\}, \\ \Sigma_m &:= \{\mathbf{a} \in \Sigma \mid m \not\leq_{\mathcal{J}} \mathbf{a}\}, \end{aligned}$$

and, using these sets, we define three languages

$$\begin{aligned} U_m &:= \bigcup_{(n, \mathbf{a}) \in P_m} L_n \mathbf{a}, \quad V_m := \bigcup_{(\mathbf{a}, n) \in S_m} \mathbf{a} L_n, \text{ and} \\ W_m &:= \Sigma_m \cup \bigcup_{(\mathbf{a}, n, \mathbf{b}) \in F_m} \mathbf{a} L_n \mathbf{b}. \end{aligned}$$

In Proposition 15, we will express L_m in terms of the languages U_m , V_m , and W_m of Definition 13. The proof of Proposition 15 will rely on a characterization of the language $\Sigma^* W_m \Sigma^*$, which we prove first, in Lemma 14.

Lemma 14. *Let $m \in M$ and $w \in \Sigma^*$. Then $m \not\leq_{\mathcal{J}} \phi(w)$ if, and only if, w has a factor in W_m .*

Proof. For the left-to-right direction, suppose $m \not\leq_{\mathcal{J}} \phi(w)$. Let v be a factor of w of minimal length such that $m \not\leq_{\mathcal{J}} \phi(v)$. We will prove that $v \in W_m$. Note that $v \neq \epsilon$, since $m \leq_{\mathcal{J}} \phi(\epsilon) = 1$. We distinguish two cases:

1. $|v| = 1$.

Write $v = \mathbf{a}$ for some $\mathbf{a} \in \Sigma$. Then $\mathbf{a} \in \Sigma_m \subseteq W_m$.

2. $|v| > 1$.

Pick $\mathbf{a}, \mathbf{b} \in \Sigma$ and $u \in \Sigma^*$ such that $v = \mathbf{a}u\mathbf{b}$. Write $n := \phi(u)$. We prove that $(\mathbf{a}, n, \mathbf{b}) \in F_m$. First, since $\mathbf{a}u$ is strictly shorter than v , we must have $m \leq_{\mathcal{J}} \phi(\mathbf{a}u) = \mathbf{a}n$. By a symmetric argument, $m \leq_{\mathcal{J}} n\mathbf{b}$. Finally, since $\phi(v) = \mathbf{a}\phi(u)\mathbf{b} = \mathbf{a}n\mathbf{b}$, we have $m \not\leq_{\mathcal{J}} \mathbf{a}n\mathbf{b}$.

For the right-to-left direction, suppose that $u \in W_m$ is a factor of w . Note that this implies $\phi(w) \leq_{\mathcal{J}} \phi(u)$ by the Preservation Lemma. We distinguish two cases:

1. The word u is equal to some letter $\mathbf{a} \in \Sigma_m$.

Since $\mathbf{a} \in \Sigma_m$, we have $m \not\leq_{\mathcal{J}} \mathbf{a} = \phi(u)$, so in particular $m \not\leq_{\mathcal{J}} \phi(w)$.

2. There exist $(\mathbf{a}, n, \mathbf{b}) \in F_m$ and $v \in L_n$ such that $u = \mathbf{a}v\mathbf{b}$.

Since $(\mathbf{a}, n, \mathbf{b}) \in F_m$, we have $m \not\leq_{\mathcal{J}} \mathbf{a}v\mathbf{b} = \phi(u)$, so in particular $m \not\leq_{\mathcal{J}} \phi(w)$. \square

Proposition 15. *Let $m \in M \setminus \{1\}$. Then*

$$L_m = (U_m \Sigma^* \cap \Sigma^* V_m) \setminus (\Sigma^* W_m \Sigma^*) .$$

Proof. Let $w \in \Sigma^*$ be arbitrary. We need to show that $\phi(w) = m$ if, and only if, w has a prefix in U_m , a suffix in V_m , and no factor in W_m .

For the left-to-right direction, suppose that $\phi(w) = m$.

To see that w has a prefix in U_m , let u be the longest prefix of w such that $\phi(u) \not\leq_{\mathcal{R}} m$. Such a prefix indeed exists, because the empty word ϵ is one: we have $\phi(\epsilon) = 1 \not\leq_{\mathcal{R}} m$ because $m \neq 1$, using Lemma 10.2. Also, since $\phi(w) = m$, the word u is a strict prefix of w . Therefore, pick $\mathbf{a} \in \Sigma$ and $v \in \Sigma^*$ such that $w = u\mathbf{a}v$, and write $n := \phi(u)$. We will show that $(u, \mathbf{a}) \in P_m$. Note that $m = \phi(u)\mathbf{a}\phi(v) \leq_{\mathcal{R}} \phi(u)\mathbf{a} = n\mathbf{a}$, and, since u is the longest, $n\mathbf{a} = \phi(u\mathbf{a}) \leq_{\mathcal{R}} m$. We conclude that $n\mathbf{a} \mathcal{R} m$. Also, by the choice of u , we have $n = \phi(u) \not\leq_{\mathcal{R}} m$.

The proof that w has a suffix in V_m is symmetric, using Lemma 10.1.

To finish the proof of the left-to-right direction, let u be any factor of w . Then $m = \phi(w) \leq_{\mathcal{J}} \phi(u)$ by the Preservation Lemma. By Lemma 14, u does not have any factor in W_m , and in particular, $u \notin W_m$.

For the right-to-left direction, suppose that w has a prefix in U_m , a suffix in V_m , and no factor in W_m . Pick $(n, \mathbf{a}) \in P_m$, $u \in L_n$, $(\mathbf{b}, p) \in S_m$, and $v \in L_p$ such that $u\mathbf{a}$ is a prefix of w and $\mathbf{b}v$ is a suffix of w . Using the Preservation Lemma, we have $\phi(w) \leq_{\mathcal{R}} n\mathbf{a} = \phi(u\mathbf{a}) \leq_{\mathcal{R}} m$. Similarly, $\phi(w) \leq_{\mathcal{L}} m$. By Lemma 14, we have $m \leq_{\mathcal{J}} \phi(w)$. Thus, by strong \mathcal{H} -triviality (Corollary 9.3), we conclude $\phi(w) = m$. \square

Proposition 16. *Let M be a finite aperiodic monoid. For any $m, n, a, b \in M$,*

1. *if $m \leq_{\mathcal{R}} na$ and $n \not\leq_{\mathcal{R}} m$, then $m <_{\mathcal{J}} n$;*
2. *If $m \leq_{\mathcal{L}} bn$ and $n \not\leq_{\mathcal{L}} m$, then $m <_{\mathcal{J}} n$;*
3. *If $m \leq_{\mathcal{J}} an, m \leq_{\mathcal{J}} nb$, and $m \not\leq_{\mathcal{J}} anb$, then $m <_{\mathcal{J}} n$.*

Proof. (1) Note that $m \leq_{\mathcal{R}} na \leq_{\mathcal{R}} n$, so in particular $m \leq_{\mathcal{J}} n$. If we would have $n \leq_{\mathcal{J}} m$, then, since $m \leq_{\mathcal{R}} n$, Corollary 9.2 implies $n \leq_{\mathcal{R}} m$, contrary to the hypothesis.

(2) is symmetric, using Corollary 9.1.

(3) First note $m \leq_{\mathcal{J}} an \leq_{\mathcal{J}} n$. Towards a contradiction, suppose $n \leq_{\mathcal{J}} m$. Then also $n \leq_{\mathcal{J}} an$, and clearly $an \leq_{\mathcal{L}} n$. Thus, $n \leq_{\mathcal{L}} an$ by Corollary 9.1. Since the preorder $\leq_{\mathcal{L}}$ is right-invariant (Exercise 5), we get $nb \leq_{\mathcal{L}} anb$, and thus in particular $nb \leq_{\mathcal{J}} anb$. Since $m \leq_{\mathcal{J}} nb$ by hypothesis, we conclude $m \leq_{\mathcal{J}} anb$, contrary to the hypothesis. \square

Proof of Theorem 11. For any $m \in M$, define $\uparrow_{\mathcal{J}} m := \{n \in M \mid m \leq_{\mathcal{J}} n\}$, and $j(m) := |\uparrow_{\mathcal{J}} m|$. We make three remarks about this definition:

1. If $m <_{\mathcal{J}} n$, then $\uparrow_{\mathcal{J}} n \subsetneq \uparrow_{\mathcal{J}} m$, so $j(n) < j(m)$.
2. By Lemma 10.3, $\uparrow_{\mathcal{J}} 1 = \{1\}$, so $j(1) = 1$.
3. If $m \neq 1$, then $j(m) > 1$ since $\{1, m\} \subseteq \uparrow_{\mathcal{J}} m$.

We now prove by induction that, for any $N \in \mathbb{N}_{\geq 1}$, if $m \in M$ is such that $j(m) = N$, then L_m is starfree.

For the base case $N = 1$, remarks 2 and 3 above show that $j(m) = 1$ implies $m = 1$, and L_1 is starfree by Proposition 12.

Now assume $N > 1$ and let $m \in M$ be such that $j(m) = N$. In particular, by remark 3 above, $m \neq 1$. Referring to Definition 13, for any $(n, \mathbf{a}) \in P_m$, $(\mathbf{a}, n) \in S_m$, and $(\mathbf{a}, n, \mathbf{b}) \in F_m$, Proposition 16 gives that $m <_{\mathcal{J}} n$, so that $j(n) < j(m)$ by remark 1 above. By the induction hypothesis, L_n is starfree in all of these cases. Thus, the languages U_m , V_m and W_m of Definition 13 are also starfree. By Proposition 15, L_m is starfree.

We conclude that L_m is starfree for every $m \in M$. For any $T \subseteq M$, we have $\phi^{-1}(T) = \bigcup_{m \in T} L_m$, so $\phi^{-1}(T)$ is starfree as well. \square

The technique used in the proof of Theorem 11 is called ‘induction on the \mathcal{J} -ordering’, and is a common tool in the theory of finite monoids, see, e.g., [2].

4 From starfree to aperiodic

Proof of Theorem 1, left-to-right direction. Starfree languages are in particular regular, so their syntactic monoids are finite. Let us call $\ell \in \mathbb{N}$ an *aperiodic index* for a language L if, for any $u \in \Sigma^*$, we have $u^\ell \sim_L u^{\ell+1}$. We will prove by induction that any starfree language L has an aperiodic index. This clearly implies that M_L is aperiodic. The induction is on the construction of the starfree language L , distinguishing cases:

1. $L = \emptyset$.

The syntactic congruence identifies all words, so 0 is an aperiodic index for L .

2. $L = \{\epsilon\}$.

The syntactic congruence identifies all non-empty words, so 1 is an aperiodic index for L .

3. $L = \{a\}$ for some $a \in \Sigma$.

The syntactic congruence has two classes: words containing a , and words not containing a . For any word $u \in \Sigma^*$, the word u contains a if, and only if, the word u^2 contains a , so 1 is an aperiodic index for L .

4. $L = K_1 \cup K_2$ for starfree K_1, K_2 .

By the induction hypothesis, pick aperiodic indices ℓ_1 and ℓ_2 for K_1 and K_2 , respectively. Define $\ell := \max\{\ell_1, \ell_2\}$. Note that ℓ is an aperiodic index for both K_1 and K_2 : indeed, for any $u \in \Sigma^*$, we have $u^\ell = u^{\ell_i} u^{\ell - \ell_i}$, and \sim_{K_i} is a congruence, from which $u^\ell \sim_{K_i} u^{\ell+1}$ follows. We show that ℓ is an aperiodic index for L . Let $u, \alpha, \beta \in \Sigma^*$, and suppose that $\alpha u^\ell \beta \in L$. Pick $i \in \{1, 2\}$ such that $\alpha u^\ell \beta \in K_i$. Then $\alpha u^{\ell+1} \beta \in K_i$ since ℓ is an aperiodic index for K_i . Thus, $\alpha u^{\ell+1} \beta \in L$. The reverse direction is similar.

5. $L = \Sigma^* \setminus K$ for starfree K .

By the induction hypothesis, pick an aperiodic index ℓ for K . The same ℓ is an aperiodic index for L , as follows directly from the definition of \sim_L .

6. $L = K_1 \cdot K_2$ for starfree K_1, K_2 .

By the induction hypothesis, pick aperiodic indices ℓ_1 and ℓ_2 for K_1 and K_2 , respectively. Define $\ell := \ell_1 + \ell_2$. We show that ℓ is an aperiodic index for L .

Let $u \in \Sigma^*$. Suppose that $u^\ell \in L$. We show that $u^{\ell+1} \in L$. Pick $v_1 \in K_1$ and $v_2 \in K_2$ such that $u^\ell = v_1 v_2$. We distinguish two cases:

(a) $|v_1| \geq \ell_1 |u|$.

In this case, we can write $v_1 = u^{\ell_1} \beta$ for some $\beta \in \Sigma^*$, and we must have $\beta v_2 = u^{\ell_2}$ since $v_1 v_2 = u^{\ell_1 + \ell_2}$ (see Figure 1). Define $v'_1 := u^{\ell_1 + 1} \beta$. Since $u^{\ell_1} \sim_{K_1} u^{\ell_1 + 1}$ by hypothesis, and $u^{\ell_1} \beta = v_1 \in K_1$, we also have $v'_1 = u^{\ell_1 + 1} \beta \in K_1$. Thus, $u^{\ell+1} = u^{\ell_1 + 1} \beta v_2 = v'_1 v_2 \in L$.

(b) $|v_1| < \ell_1 |u|$.

In this case, $|v_2| = \ell |u| - |v_1| > (\ell - \ell_1) |u| = \ell_2 |u|$, so we can proceed as in the previous case, by symmetry.

In order to prove that $u^{\ell+1} \in L$ implies $u^\ell \in L$, we proceed similarly, writing $u^{\ell+1} = v_1 v_2$ with $v_1 \in K_1$ and $v_2 \in K_2$, and distinguishing cases:

(a) $|v_1| \geq (\ell_1 + 1) |u|$.

Write $v_1 = u^{\ell_1 + 1} \beta$ where $\beta \in \Sigma^*$ and $\beta v_2 = u^{\ell_2}$. Define $v'_1 := u^{\ell_1} \beta$. As before, we have $v'_1 \in K_1$. Thus, $u^\ell = v'_1 v_2 \in L$.

(b) $|v_1| < (\ell_1 + 1) |u|$.

In this case $|v_2| = (\ell + 1) |u| - |v_1| > (\ell - \ell_1) |u| = \ell_2 |u|$, and we proceed as in the previous case, by symmetry. \square

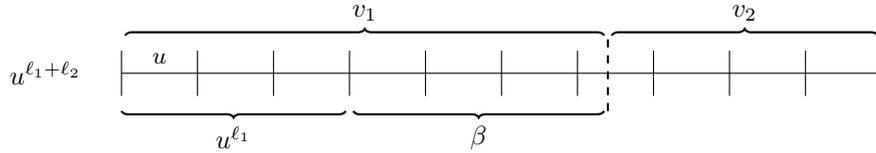


Figure 1: Illustration of case 6a in the proof of Theorem 1, left-to-right direction.

References

- [1] Olivier Carton, *Langages formels, Calculabilité et Complexité*, seconde édition, Éditions Vuibert, 2014. Chapter 1 available online at <https://www.irif.fr/~carton/Lfcc/chap1.pdf>.
- [2] Thomas Colcombet, “Green’s Relations and their Use in Automata Theory”. In: *Language and Automata Theory and Applications - 5th International Conference* (Adrian Horia Dediu, Shunsuke Inenaga, and Carlos Martín-Vide, editors), *Lect. Notes Comput. Sci.* 6638, Springer, pp. 1-21 (2011). doi:10.1007/978-3-642-21254-3_1.

- [3] Jean-Éric Pin, *Mathematical Foundations of Automata Theory*, version of 24 March 2025. Available online at <https://www.irif.fr/~jep/PDF/MPRI/MPRI.pdf>.
- [4] Howard Straubing, “First-order logic and aperiodic languages: A revisionist history.” *ACM SIGLOG News*, vol. 5, no. 3, pp. 4-20 (2018). doi:10.1145/3242953.3242956.