On sheaves and duality

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Contents

Introduction

This thesis is concerned with sheaves and duality, using order completions as an essential tool. The interplay between duality, sheaves and order completions will be a recurring theme in this thesis. In this introduction, we discuss our motivation for studying these topics, and give an overview of the chapters to come.

Two approaches to logic

Logic provided the main motivation and inspiration for many of the topics that are addressed in this thesis. In the broadest possible terms, logic studies the structure of arguments. To do so, several kinds of mathematical methods can be employed, which we will broadly divide in *algebraic* and *spatial* approaches to logic.

An *algebraic approach* to logic starts from the inherent syntactic structure of logical arguments. Algebra can be a useful tool to abstractly study such syntactic structure. For example, just as commutative rings are an abstraction of the integers with the operations of addition, multiplication, and subtraction, *Boolean algebras* are an abstraction of propositional sentences with the operations of classical disjunction ('or'), conjunction ('and') and negation ('not'). Algebra has a great generalizing power; for example, if, instead of integers, one wants to study matrices with integer entries, then one can still use the framework of rings, dropping the axiom of commutativity. Similarly, if one is interested in logics with quantifiers, modalities, or non-classical operations, then one can consider generalizations of Boolean algebras. In this approach to logic, the relation of logical derivability is formalized as a partial order in the algebra, and logical equivalence thus corresponds to equality.

On the other hand, a *spatial approach* to logic starts from a collection, or space, of 'models' that can interpret logical formulas. The 'meaning' of a logical formula is then defined using this space. For example, in classical propositional logic, a point in the space is an assignment of truth values to propositional variables. Any propositional formula then defines the subset of points (models) where that formula is true. In this spatial approach, two logical formulas are considered equivalent if they are true in exactly the same models.

Duality in logic: an example

Duality provides a link between the algebraic and spatial approaches to logic. To illustrate the use of duality in logic, we now briefly sketch a proof of the completeness theorem for first-order logic. In the proof of that theorem, the crux of the argument is to show that, if a first-order sentence φ is not syntactically derivable from a set of first-order sentences Γ, then there exists a first-order model in which all sentences in Γ are true, but φ is not.

Duality theory for Boolean algebras, as developed in the 1930's by M. H. Stone, says that any Boolean algebra can be represented as the collection of clopen subsets of some compact Hausdorff topological space. In particular, we can apply this fact to the collection of first-order sentences, considered up to provable equivalence. To any point x of the corresponding topological space, one may associate the set of first-order sentences which (under Stone's representation) contain the point x. The sets of sentences obtained in this manner are known as complete consistent theories, or ultrafilters. Any such ultrafilter can be used to canonically define a first-order model, M_x. This model M_x will be especially useful if the first-order sentences that are true in M_x are exactly the sentences which contain the point x. Call a point x *special* if its associated model has this useful property. Since Stone's space is compact and Hausdorff, one may use *Baire category theorem* from general topology to prove that the special points are dense in any closed subspace. Now, if φ is a first-order sentence that is not syntactically derivable from a set of sentences Γ, then the intersection of the sentences in Γ is closed and not contained in φ; by denseness, there is a special point x in this closed set which is not in φ. The associated model M_x is the first-order model that we needed to construct.[1]

The proof outlined in the previous paragraph follows a general scheme, which may be described as follows. A problem from logic (finding a countermodel for a non-derivable sentence) is solved by translating it into a topological question (does there exist a 'special point'?), which can be answered using the well-developed theory of general topology (Baire category theorem). Duality is the general mechanism which enables one to translate back and forth between algebraic and spatial approaches to logic. Mathematically speaking, a *duality* is a contravariant categorical equivalence between a category of algebras and a category of spaces. This means that any algebra has a space associated to it, and vice versa, in such a way that maps between algebras correspond to maps between the associated spaces, in the reverse direction. We will give more precise mathematical

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[1]The completeness of first-order logic was first proved by Gödel [69]. The proof that we outlined in this paragraph was first given by H. Rasiowa and R. Sikorski in [129]; also see the classical reference [130].

background on Stone duality in Section 1.1 of this thesis.

In the above example, the collection of models M_x associated to the (special) points x in the space played a crucial role. This collection of models M_x can be made in such a way that it 'varies continuously' in the variable x. Sheaves, the other focal point of this thesis, are designed precisely to deal with structures that vary continuously over a topological space. These observations form the basis of the sheaf-theoretic, categorical approach to first-order logic.

Using order completions

Stone's original duality theory was concerned with Boolean algebras, which model the calculus of formulas in classical propositional logic. In the above example, Stone duality was used to prove a basic theorem in classical first-order logic. Many other kinds of logic, including modal, intuitionistic, substructural, and multi-valued logic, can be studied using generalizations of Stone duality. Priestley duality is one such generalization: it deals with distributive lattices, i.e., the algebraic structures corresponding the negation-free fragment of propositional logic, cf. Figure 1 for an example. Often, one may want to enrich the algebraic theory of distributive lattices with operations, such as implication. This can lead, for example, to a definition of Heyting algebras, the algebraic structures for Brouwer's intuitionistic logic. If one imposes slightly different axioms for implication, one may arrive at algebraic structures for other logics. For example, the natural algebraic structures for infinite-valued Łukaciewicz logic, MV-algebras, are distributive lattices enriched with operations of addition and subtraction. Chapter 4 of this thesis is devoted to sheaf representations and duality theory for MV-algebras. For a slightly different, but related, example, algebraic structures for modal logic are obtained by adding so-called modality operators to distributive lattices or Boolean algebras.

Thus, from the point of view of logic, it is useful to have a duality theory that also accounts for additional operations on Boolean algebras and lattices. For example, if one adds an operation to a Boolean algebra, one may wonder to what kind of structure this corresponds on the space dual to the algebra. Order completions, and canonical extensions in particular, have been useful to answer questions of this kind. Canonical extensions provide an algebraic perspective on duality theory; we will briefly sketch this idea, referring to Section 1.2 for more mathematical details. Broadly speaking, the canonical extension of an algebraic structure is a *complete* extension of it, in the sense that arbitrary limits exist, which *preserves the finitary structure*. For example, the canonical extension of a Boolean algebra is the power set algebra of the Stone's representing space. Note that, in the above proof of

the completeness theorem of first-order logic, we already implicitly used the canonical extension, in order to make statements such as "the set of first-order formulas which contain a point x". Canonical extensions also exist for lattices in general, which makes the scope of the theory very wide. We will use canonical extensions for lattices to study topological duality for lattices in Chapter 6 of this thesis. A further advantage of order completions is that they often allow for a relatively simple treatment of *morphisms* between algebras. This is both the case for canonical extensions, as already mentioned above, and for a different kind of order completion that we call the *frame completion* here. We will discuss the relationships between Stone duality and order completions in more detail in Chapter 1. Frame completions will play a role in Chapter 3, and also, alongside canonical extensions, in Chapter 2.
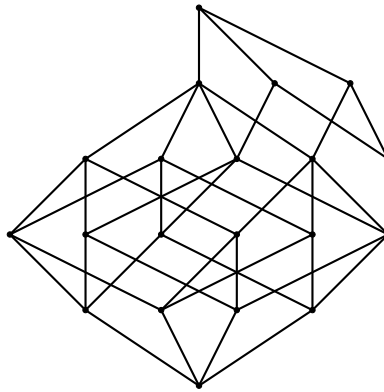
Figure 1: An example of a distributive lattice A.

Sheaves and duality

So far, we have discussed the use of duality and order completions in logic, but we have mentioned sheaves only briefly. Sheaves will appear in chapters 3, 4 and 5 of this thesis. Here, we will sketch the basic relationship between sheaves and duality theory in the context of finite lattices.

Consider the distributive lattice A in Figure 1. With the purpose of gaining a better understanding of A, one may be interested in direct product decompositions of A, i.e., collections of lattices $A_1, \ldots, A_n$ such that $A \cong A_1 \times \cdots \times A_n$. The Priestley dual space of A characterizes all such decompositions: it follows from (finite) Priestley duality that product decompositions of A correspond to disjoint sum decompositions of the dual space of A. Because this example is finite, the dual space of A has a trivial topology, but a non-trivial order, depicted in Figure 2. One may now read off all possible product decompositions of A from the dual space. For ex-

ample, consider the decomposition of the dual space into two pieces, one piece containing the two points on the left, the other containing the three points on the right. This decomposition corresponds to a representation of A as the product of the two lattices A_0 and A_1 in Figure 3.

Figure 2: The dual space of A.
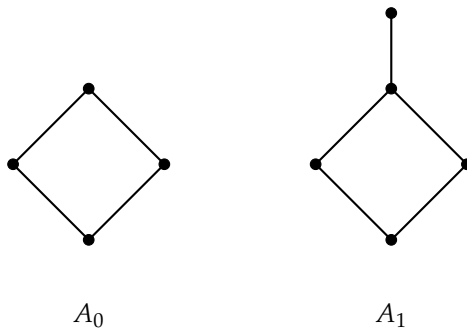
A_0 $\qquad\qquad\qquad\qquad$ A_1

Figure 3: Distributive lattices A_0, A_1 for which $A \cong A_0 \times A_1$.

In the infinite case, it will usually be impossible to find full product decompositions such as the one in the above example. However, one can often still find an embedding of A into a product of simpler structures $(A_i)_{i \in I}$, say. Elements of A then correspond to certain elements of the full product $P := \prod_{i \in I} A_i$. The embedding of A into the product P is called a sheaf representation if there exists a "topological description" of those elements of P which come from elements of A. We now broadly explain the meaning of the phrase "topological description" in the previous sentence. First note that an element of the full direct product P is a function s from I to the union of the lattices A_i with the property that $s(i) \in A_i$ for all $i \in I$; such functions are called (discrete) *global sections*. In particular, elements of A give rise to global sections. Suppose further that both the index set I and the set-theoretic union of the A_i are equipped with a topology. Under these assumptions, the embedding of A into P is a *sheaf representation* if the global sections coming from A are exactly the global sections which are continuous with respect to these topologies. In Chapter 3 of this thesis, we will show that sheaf representations of a distributive lattices still correspond to certain decompositions of the Priestley dual space of A, as in the finite example that we gave here.

Overview of the thesis
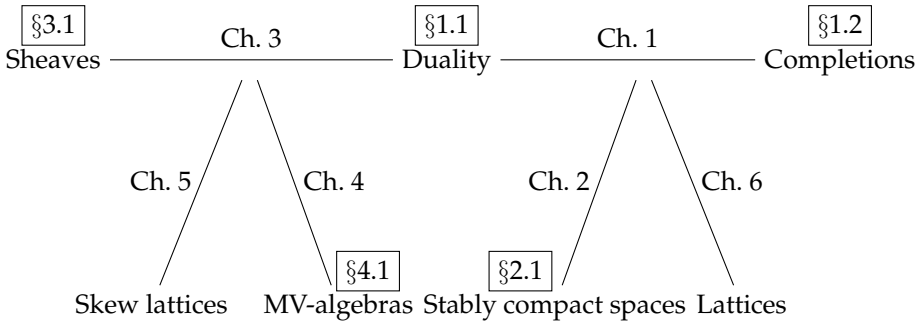
Figure 4: In this diagram, boxes refer to the sections where the relevant pre-liminaries can be found, while lines indicate the central topics per chapter; e.g., chapter 4 discusses duality and sheaves for MV-algebras.

In **Chapter 1**, we recall the relevant mathematical background on Stone duality and order completions, in particular canonical extensions and frame completions. While doing so, we pay particular attention to the intimate connections between duality and completions.

Traditionally, Stone-type dualities and canonical extensions have been most usefully applied to classes of spaces which have a canonical basis for the topology. We will show in **Chapter 2** that the connection between Stone duality and canonical extensions still exists in the wider context of *stably compact spaces*, where a canonical choice of basis is not available. Stably compact spaces form a robust class which is large enough to include all spaces that play a role in the existing theory of Stone duality and canonical extensions, but also, for example, compact Hausdorff spaces, which are out of the scope of Stone's original duality.

Sheaves will enter the mathematical stage in **Chapter 3**, where we study Priestley duality for sheaf representations of distributive lattices. The two main ingredients for doing so are the theory of stably compact spaces studied in Chapter 2 and the perspective on duality provided by frame completions in Chapter 1. These two strands combine to provide a complete dual characterization of sheaves of distributive lattices that are 'flasque on a basis', a generalization of the well-known notion of 'flasque'.

The duality theory for flasque sheaves on a basis is applied to the setting of MV-algebras, or equivalently, unital lattice-ordered abelian groups, in **Chapter 4**. MV-algebras provide the algebraic semantics for Łukasiewicz infinite-valued propositional logic, thus playing an analogous role to that of

Boolean algebras in classical propositional logic. We show in particular that two known sheaf representations of MV-algebras are both consequences of the results proved in Chapter 3. Moreover, these methods also allow us to prove an MV-algebraic generalization of a classical theorem by Kaplansky, which says that two compact Hausdorff spaces are homeomorphic if the lattices of continuous $[0, 1]$-valued functions on the spaces are isomorphic.

In **Chapter 5**, we study a non-commutative generalization of lattices, skew lattices, and show how they capture certain algebraic structure intrinsic to sheaves. A typical example of a skew lattice is the collection of partial functions from a set X to a set Y, equipped with two binary operations called *restriction* and *override*. Here, the restriction of f by g is defined as the function which takes the value of f only if both f and g are defined, and the override of f by g is the function which takes the value of g if it is defined, and the value of f otherwise. These operations are clearly non-commutative. Equipping X with a topology and letting the co-domain Y vary over X, this construction can be generalized to local sections of sheaves over Priestley spaces. The main result of Chapter 5 establishes a duality between skew distributive lattices and sheaves of sets over Priestley spaces, thus generalizing Priestley duality to a non-commutative setting.

In the final chapter of this thesis, **Chapter 6**, we continue our study of generalizations of Stone-Priestley duality for distributive lattices. We come back to the connection between duality and order completions that we pointed out in Chapter 1, and use it to study topological duality for bounded lattices in general, dropping the axiom of distributivity. While doing so, we develop the *distributive envelope* for a lattice, an order-theoretical construction that could be of interest independent from topological duality.

Relevant publications

- Chapter 2 is a modified version of the publication [72].

- The results in Chapter 3 will be the topic of [63].

- Chapter 4 is based on the results in [65], currently under submission.

- Chapter 5 is a modified version of the publication [8].

- Chapter 6 is a modified version of the publication [64].

- We refer the interested reader to the publication [34], which was also written as part of the author's PhD project. Since [34] is thematically less closely related to our other publications listed above, we chose not to include it in this thesis, also in order to avoid an overly long document. A version of the paper [34] can also be found in [32, Chapter 2].

Chapter 1. Stone duality and completions

In this preliminary chapter, we will first recall the basics of duality theory for Boolean algebras and distributive lattices. We will then discuss the relation between duality theory and order completions, with a particular focus on canonical extensions and frame completions.

1.1. Stone duality and its successors

The starting point of duality theory was a representation of Boolean algebras by Stone, which he called the "perfect representation" ([137, Section 4.3]). This representation confirmed the idea that the axioms for Boolean algebras precisely captured the "algebra of classes", that is, the structure of the collection of subsets of a given set. Moreover, it provided a first connection between Boolean algebras and certain topological spaces. In modern terms, Stone's representation theorem can be stated as follows.

Theorem 1.1.1 (Theorem 67, [137]). *For every Boolean algebra B, there exists a unique embedding of Boolean algebras $\eta_B : B \hookrightarrow \mathcal{P}(X)$ for which the topology, τ, on X that is generated by the image of η_B is compact and Hausdorff.*

We highlight a few aspects of the proof of this theorem, which also allows us to fix some notation. One may first show by topological arguments that if η_B is an embedding such as in the statement, then (X, τ) is homeomorphic to the space of homomorphisms from B to $\mathbf{2}$, the two-element Boolean algebra. Here, the topology on the space of homomorphisms is inherited from the product topology on $\mathbf{2}^B$, where $\mathbf{2}$ has the discrete topology. Therefore, to prove that η_B exists, one may now *define* X to be the space of homomorphisms $B \to \mathbf{2}$ and $\eta_B : B \hookrightarrow \mathcal{P}(X)$ the map that sends b to the set $\hat{b} := \{x \in X \mid x(b) = 1\}$. The key part of the proof is to show that η_B is injective; this is essentially the content of Stone's prime ideal theorem [137, Theorem 64], also known as the ultrafilter lemma, which uses the axiom of choice. For the proof of the ultrafilter lemma in modern notation, see [36, Theorem 10.17], or see the proof of Theorem 2.5.2 in Chapter 2 of this thesis, which generalizes, and was inspired by, the ultrafilter lemma. The topological space (X, τ) that is uniquely associated to a Boolean algebra B by Theorem 1.1.1 is called the (Stone) *dual space* of B, and will also be denoted by B_*.

Remark 1.2.2. Let X be the dual space of a Boolean algebra B. Notice that the subsets of X of the form $\hat{b}$, for $b \in B$, are *clopen* (closed and open) in

the topology on X. Using compactness of X, one may also show that, conversely, any clopen set is of the form $\widehat{b}$ for some $b \in B$. Thus, the clopen sets in X are exactly the sets of the form $\widehat{b}$, for $b \in B$, and these form a basis for the topology.

Definition 1.1.3. A topological space is *Boolean*[1] if it is compact, Hausdorff, and the clopen sets form a basis.

Theorem 1.1.4 (Stone duality for Boolean algebras [137]). *The category of Boolean algebras with homomorphisms is dually equivalent to the category of Boolean spaces with continuous functions.*

We briefly recall how the dual equivalence in this theorem works. If X is a Boolean space, then the collection of clopen subsets forms a Boolean algebra, that we will denote by X^*. Moreover, any continuous function $f : X \to Y$ yields a homomorphism of Boolean algebras $f^* := f^{-1} : Y^* \to X^*$. Note that $(-)^*$ turns the order of arrows around and preserves composition and identity: it is a *contravariant functor*. In the other direction, there is a contravariant functor $(-)_*$ which sends a Boolean algebra to its dual space B_*, and a homomorphism $h : B \to C$ to a continuous function $h_* : C_* \to B_*$ defined by "precomposing with h", using that the points of the dual space are themselves homomorphisms to **2**. To prove Theorem 1.1.4, one now shows that the functors are mutually inverse up to natural isomorphism. For more details, cf., e.g., [93, Sections 3.7 and 3.8].

Remark 1.1.5. A homomorphism x from a Boolean algebra B into **2** can alternatively be described by the *prime filter* $F_x := x^{-1}(1)$ or the *prime ideal* $I_x := x^{-1}(0)$.

Stone published the theory described above in 1936 [137]. Little more than a year later, he published a generalization of these results to distributive lattices [138], that we will recall in Theorem 1.1.7 below. Birkhoff had already proved a representation theorem for distributive lattices in 1933 [13], but had not been concerned with the topological aspects of the theory. The construction of the dual space of a distributive lattice is completely analogous to the Boolean case. However, since Remark 1.1.2 only applies to Boolean algebras, the sets which represent the lattice are no longer clopen, but only compact-open. Thus, the dual spaces of distributive lattices no longer need to be Boolean. The definition of the relevant class of spaces, that we recall below, is more involved than that of Boolean spaces. It must have looked even less natural at the time of its publication, when non-Hausdorff spaces

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[1]We follow Stone's terminology [136]. Boolean spaces are also called *Stone spaces* in the literature (e.g. [82]), but we choose to avoid that name, as it has also been used for other classes of spaces, e.g. in [4].

were hardly ever considered; even Stone himself seems to have been some-what dissatisfied with the definition, cf. the introduction of [138].

Definition 1.1.6. A topological space is *sober* if every non-empty closed set that can not be written as the union of two smaller closed sets is the closure of a single point. A topological space is *spectral*[2] if it is T_0, sober, and the collection of compact-open sets forms a basis for the open sets that is closed under finite intersections.

Observe that a spectral space is Hausdorff if, and only if, it is Boolean. There is a particular subclass of continuous maps between spectral spaces which correspond to distributive lattice homomorphisms. We say that a function $f : X \to Y$ between spectral spaces is *perfect* if the inverse image under f of any compact-open set in Y is compact-open in X. Notice that a perfect map is in particular continuous, but that the converse does not hold. (We will reconsider the full class of continuous maps between spectral spaces in Section 2.2 in Chapter 2 below.) With these definitions, the following may be proved in an analogous way to Theorem 1.1.4:

Theorem 1.1.7 (Stone duality for distributive lattices [138]). *The category of distributive lattices with homomorphisms is dually equivalent to the category of spectral spaces with perfect functions.*

In this duality, a spectral space X corresponds to its distributive lattice X^* of compact-open subsets. Conversely, the *Stone dual space* D_* of a distributive lattice D is defined as the set of homomorphisms $D \to \mathbf{2}$, equipped with the topology generated by the basis $\{\widehat{d} \mid d \in D\}$, where $\widehat{d}$ is the set of homomorphisms which send d to 1.

Example 1.1.8. The prime ideal spectrum of a commutative unital ring R, equipped with the Zariski topology, is a spectral space X. The distributive lattice corresponding to the spectral space X can be constructed directly from the ring R, namely as the lattice of finitely generated *radical ideals*. Hochster [80] proved that any spectral space arises as the prime ideal spectrum of some commutative unital ring. Also see Banaschewski [6].

Priestley [127], in the paper based on her PhD thesis, gives a considerably cleaner presentation of duality for distributive lattices. Priestley's duality is based on a category of *ordered* topological spaces, which goes back to Nachbin [122], also see Section 2.1 below.

Definition 1.1.9. A *Priestley space*, or *compact totally order-disconnected space*, is a tuple $(X, \tau, \leq)$, where τ is a compact topology on X and $\leq$ is a partial

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[2]The name spectral space seems due to Hochster [80]. These spaces have also been called Stone spaces [4] or coherent spaces [82] in the literature.

order on X, such that, for any $x, y \in X$, if $x \nleq y$, then there exists a clopen downset K such that $y \in K$ and $x \notin K$.

Note that the condition that Boolean spaces have a basis of clopen sets can be reformulated as: if $x \neq y$ then there is a clopen set K containing y and not containing x. Therefore, the topological space underlying a Priestley space is always Boolean, and Boolean spaces correspond precisely to Priestley spaces in which the order is trivial.

Theorem 1.1.10 ([127]). *The category of distributive lattices with homomorphisms is dually equivalent to the category of Priestley spaces with continuous order-preserving functions.*

In this dual equivalence, given a Priestley space X, the associated distributive lattice is the lattice of clopen downsets of X, that we shall also[3] denote by X^*. The Priestley dual space D_* of a distributive lattice D can be defined as follows. The points of D_* are the same as in Stone's duality, namely the homomorphisms $D \to \mathbf{2}$. The key idea is now to use the sets $\hat{d}$ and *their complements* to generate a topology τ^p on D_*. This topology refines Stone's topology on the dual space of a distributive lattice (also see Example 2.1.11.2 in Chapter 2 for a more precise statement of the relation between the two topologies). The order on D_* is the reverse of the pointwise order, that is, $x \leq x'$ if, and only if, $x'(d) = 1$ implies $x(d) = 1$ for all $d \in D$. One may then show that D_* is a Priestley space and that any distributive lattice D is isomorphic to the clopen downsets of its Priestley dual space D_*. The treatment of morphisms is analogous to that in Stone duality, cf. the remarks after Theorem 1.1.4 above.

We end this section by giving two example applications of Priestley duality, each of which will be used later in this thesis.

Example 1.1.11 (Lattice quotients and closed subspaces). By Priestley duality, epimorphisms in the category of distributive lattices correspond to monomorphisms in the category of Priestley spaces. In particular, let D be a distributive lattice with Priestley dual space X. The epimorphisms whose domain is D can be described, up to isomorphism, by congruences on D. The monomorphisms whose codomain is X can be described, up to isomorphism, by closed subspaces of X. This argument shows that there is a bijective correspondence between congruences on D and closed subspaces of X, which is moreover order-reversing, by naturality of the dual

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[3]Note that we use the same notation, $(-)^*$, for Stone's and Priestley's duals. This can not lead to confusion, as a space X which is simultaneously a Priestley space and a spectral space must be Boolean, and the two definitions of X^* coincide in this case. Similar remarks apply to the notation $(-)_*$.

equivalence. We now give a concrete, and slightly more general, description of this correspondence between congruences on a distributive lattice and closed subspaces of its Priestley dual space.

Proposition 1.1.12. *Let D be a distributive lattice and let X be its Priestley dual space. There is a contravariant Galois connection between the subsets of $D \times D$ and the subsets of X, given by*

$$\vartheta \subseteq D \times D \quad \longmapsto \quad \varphi(\vartheta) := \left\{ x \in X \mid \forall (a, b) \in \vartheta \ (x \in \widehat{a} \leftrightarrow x \in \widehat{b}) \right\},$$

$$S \subseteq X \quad \longmapsto \quad \psi(S) := \left\{ (a, b) \in D \times D \mid \forall x \in S \ (x \in \widehat{a} \leftrightarrow x \in \widehat{b}) \right\}.$$

Moreover, for $\vartheta \subseteq D \times D$, we have $\vartheta = \varphi(\psi(\vartheta))$ if, and only if, ϑ is a distributive lattice congruence. Also, for $S \subseteq X$, we have $S = \psi(\varphi(S))$ if, and only if, S is closed in X. In particular, φ and ψ establish an isomorphism between the poset of congruences of D, ordered by inclusion, and the poset of closed subsets of X, ordered by reverse inclusion.

Proof. See, e.g., [128, Lemma 12] or [36, 11.32]. □

Example 1.1.13 (Center and Booleanization of a distributive lattice). Let I denote the full and faithful inclusion functor of the category of Boolean algebras in the category of distributive lattices. The functor I has a right adjoint, which is given on objects by taking the *center* of a distributive lattice D, i.e., the Boolean subalgebra consisting of the complemented elements of D. The functor I also has a left adjoint, called *free Boolean extension* or *Booleanization*, that we describe now.

Proposition 1.1.14. *Let D be a distributive lattice. The following Boolean algebras are isomorphic, and both are Booleanizations of D:*

1. *the Boolean algebra of clopen subsets of the Priestley dual space D_* of D;*

2. *the center of the congruence lattice of D.*

Proof. Observe that if $D \hookrightarrow B$ is a lattice embedding of D into *any* Boolean algebra B, then the Boolean algebra generated by the image of D is isomorphic to the Booleanization of D. It thus suffices to show that any clopen subset of D_* is a Boolean combination of clopen downsets. This follows by a standard argument from the definition of the topology and the fact that D_* is compact. Now observe that the isomorphism between the congruence lattice of D and the closed subsets of D_* (Proposition 1.1.12) restricts to an isomorphism between the centers of these two lattices. □

We remark that the isomorphism between (1) and (2) in Proposition 1.1.14 can be restricted further to the *factor congruences* of D, i.e., those congruences in the center which moreover permute with their complement. A simple argument shows that the factor congruences of D correspond to the clopen subsets of D_* which are both downsets and upsets.

1.2. Order completions: duality in algebraic form

Stone's duality provided a framework for studying Boolean algebras and the homomorphisms between them. However, in the years after Stone's work, the need arose to study functions on Boolean algebras which do not preserve all the Boolean operations, but, e.g., only preserve $\vee$ and 0, but not necessarily $\wedge$ and 1. Such functions are called (unary) *operators* on Boolean algebras. The interest in operators on Boolean algebras motivated Jónsson and Tarski [84, 85] to give an algebraic formulation of Stone's duality for Boolean algebras.

Theorem 1.2.1 ([84], Theorems 1.22–1.24). *For any Boolean algebra B, there exists a unique embedding of Boolean algebras $B \hookrightarrow C$ into a complete Boolean algebra C satisfying the following two properties:*

1. *for any $u \in C$, $u = \bigvee\{\bigwedge S \mid S \subseteq B, \bigwedge S \leq u\}$;*

2. *for any subset $T \subseteq B$, if $1 \leq \bigvee T$ in C, then there exists a finite subset $T' \subseteq T$ such that $1 \leq \bigvee T'$ in B.*

The unique complete Boolean algebra C and the embedding $B \hookrightarrow C$ are called the *canonical extension* of B and denoted by $B \hookrightarrow B^\delta$. There is an obvious structural similarity between this theorem and Stone's representation theorem (Theorem 1.1.1). Indeed, it is not hard to verify that the embedding $B \hookrightarrow \mathcal{P}(X)$, with X the Stone dual space of B, satisfies conditions (1) and (2) in this theorem. However, Theorem 1.2.1 can also be proved by a purely algebraic construction, without using the axiom of choice[4]. One may then *deduce* Stone's theorem from Theorem 1.2.1 by observing that, using the axiom of choice, the canonical extension of a Boolean algebra is isomorphic to the power set algebra of its set of atoms. Thus, *completion describes duality in algebraic form*.

Gehrke and Jónsson [56] generalized Theorem 1.2.1 and the ensuing theory to the setting of distributive lattices, and Gehrke and Harding [54] subsequently observed that essentially the same methods work for lattices which are not necessarily distributive. In Chapter 2 of this thesis, we will further generalize canonical extensions to proximity lattices. We now recall the relevant definitions and facts from [54].

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[4]We will give one such proof, in a more general setting, in Section 2.4 (p. 41 and further).

Definition 1.2.2. Let L be a lattice. A *canonical extension* of L is an embedding $e : L \hookrightarrow C$ of L into a complete lattice C satisfying

1. for all $u \in C$,

$$\bigvee \left\{ \bigwedge e[S] \mid S \subseteq L, \bigwedge e[S] \leq u \right\} = u = \bigwedge \left\{ \bigvee e[T] \mid T \subseteq L, u \leq \bigvee e[T] \right\};$$

2. for all $S, T \subseteq L$, if $\bigwedge e[S] \leq \bigvee e[T]$ in C, then there are finite $S' \subseteq S$ and $T' \subseteq T$ such that $\bigwedge S' \leq \bigvee T'$ in L.

The first property is commonly referred to as *denseness*, the second as *compactness*.

Theorem 1.2.3. *Let L be a lattice. There exists a canonical extension $e : L \hookrightarrow C$. Moreover, if $e : L \hookrightarrow C$ and $e' : L \hookrightarrow C'$ are canonical extensions of L, then there is a complete lattice isomorphism $\varphi : C \to C'$ such that $\varphi \circ e = e'$.*

Proof. See [54, Prop. 2.6 and 2.7]. □

As is common in the literature and justified by this theorem, we will speak of *the* canonical extension of a lattice L and we will denote it by L^δ. Often, with the exception of Chapter 2, we will omit reference to the embedding e, and regard L as a sublattice of L^δ. The join-closure of L inside L^δ is isomorphic to the ideal completion (or *frame completion*, cf. Example 1.2.8) of L, and its elements are therefore known as the *ideal elements* of L^δ. Similarly, the closure of L under infinite meet inside L^δ is isomorphic to the filter completion or dual frame completion of L, and is the set of *filter elements* of L^δ. The elements of L are characterized in L^δ as exactly those which are both filter and ideal elements. See [54, Lemma 3.3] for proofs of the facts mentioned in this paragraph.

Again, using the axiom of choice, one can deduce that the canonical extension of a lattice 'has enough points':

Proposition 1.2.4 (Canonical extensions are perfect lattices)**.** *Let L be a lattice. The set of completely join-irreducible elements $J^\infty(L^\delta)$ of the canonical extension $\bigvee$-generates L^δ, and the set of completely meet-irreducible elements $M^\infty(L^\delta)$ of the canonical extension $\bigwedge$-generates L^δ.*

Proof. See [54, Lemma 3.4]. □

Example 1.2.5. Notice that any completely join-irreducible element is a filter element. If D is a distributive lattice, then the set $J^\infty(D^\delta)$, ordered by the restriction of the order of D^δ, is isomorphic to the partially ordered set of prime filters of D, ordered by reverse inclusion. Using Remark 1.1.5, which

equally applies to distributive lattices, this is the partially ordered set underlying the Priestley dual space D_* of D. Under this view, the set $\widehat{d}$, for $d \in D$, is simply the intersection of the downset of d in D^δ with $J^\infty(D^\delta)$. One may then show that the canonical extension $D \hookrightarrow D^\delta$ is, up to isomorphism, the map $d \mapsto \widehat{d}$ from D to the downsets of the Priestley dual space D_*, cf. [54, Remark 2.10].

The above example motivated the work in Chapter 6, where we will study topological duality for lattices in general, using canonical extensions.
We have already emphasized that canonical extensions provide an algebraic view of duality. Recall that the goal of Jónsson and Tarski [84] was to study generalizations of homomorphisms between Boolean algebras. In fact, *any* function between lattices can be extended to the canonical extension in a fairly straightforward manner. We will study extensions of operations in the more general context of proximity lattices in Chapter 2, and we will use the theory of extensions of operators on distributive lattices in Chapter 4. We defer the statement of the relevant definitions to those chapters.
There is a different algebraic approach to duality theory, which uses *frames* (or *locales*). The idea in this field of research is similar to that of canonical extensions, namely to abstract away from the underlying set of points, thus avoiding the use of choice principles. We now recall the basics of this theory, referring to, e.g., [124] or [82, Ch. II] for more details.

Definition 1.2.6. A *frame* is a complete lattice L such that, for any $a \in L$ and $B \subseteq L$, $a \wedge (\bigvee B) = \bigvee_{b \in B}(a \wedge b)$. A *frame homomorphism* is a function between frames which preserves finite meets and arbitrary joins.

For any topological space X, its lattice of open sets, $\Omega(X)$, forms a frame. A continuous function $X \to Y$ yields a frame homomorphism $\Omega(Y) \to \Omega(X)$, by taking inverse image. This defines a contravariant functor from topological spaces to frames. In an analogous way to Stone duality, one can also associate a space to any frame: for L a frame, let $\mathbf{pt}(L)$ denote the set of frame homomorphisms from L to $\mathbf{2}$. Frame homomorphisms into $\mathbf{2}$ are called *points* of the frame L, and can alternatively be described as the $\wedge$-irreducible elements of L. As in Stone duality, elements a of the frame yield subsets $\widehat{a}$ of $\mathbf{pt}(L)$, which can be used to generate a topology on $\mathbf{pt}(L)$. Note that $\mathbf{pt}$ extends to a functor from frames to topological spaces in the same way as Stone's functor $(-)_*$. There is, however, an important difference with Stone duality. The functors Ω and $\mathbf{pt}$ still form a dual *adjunction* between the category of topological spaces and the category of frames [124, Proposition 7]. However, the two functors do not form a dual equivalence between these two categories [124, p. 4]. By general category theory,

any dual adjunction between categories can be restricted to a dual equiva-
lence between (possibly empty) full subcategories. In particular, the dual
adjunction $(\Omega, \mathbf{pt})$ yields a dual equivalence between the following two
(non-trivial) categories of frames and spaces. Call a frame *spatial* if it is
isomorphic to a frame of the form $\Omega(X)$ for some topological space X (cf.,
e.g., [124, Proposition 5] for an equivalent characterization of spatiality). A
T_0 space is homeomorphic to $\mathbf{pt}(L)$ for some frame L if, and only if it is *sober*
(cf. Definition 1.1.6 above), or "*plein*" (French: full), as it was called in [124].

Theorem 1.2.7 ([124], Corollaire after Proposition 7)**.** *The functors Ω and* $\mathbf{pt}$
form a dual equivalence between the category of spatial frames and the category of
sober spaces.

Example 1.2.8. Let X be a spectral space and let $D := X^*$ be its distribu-
tive lattice of compact-open sets. The frame of open sets $\Omega(X)$ is the *frame*
completion of D, i.e., the image of D under the left adjoint to the inclusion of
the category of frames into the category of distributive lattices. Since X is
sober, it is isomorphic to the space of points of the frame completion of D.
Thus, the Stone dual space X of a distributive lattice D may be alternatively
described as the space of points of the frame completion of D.

Comparing Examples 1.2.5 and 1.2.8, we see that both the canonical ex-
tension and the frame completion can be used to construct dual spaces for
distributive lattices. We will introduce yet another "enveloping" construc-
tion, for lattices, in Chapter 6, and use it to study topological duality for
lattices in general. Also note that Booleanization (Example 1.1.13) is an en-
veloping construction from which the Priestley dual space of a distributive
lattice can be reconstructed, using Stone's duality for Boolean algebras [51,
Section 6].
The connection between the canonical-extension-approach and the frame-
theoretic approach to duality seems an interesting one to explore further;
we mention two recent works in this direction. Erné [42, Section 9] con-
structed a "canonical envelope" for spatial preframes. It is not hard to see
that, in the case where the preframe is the frame completion of a distribu-
tive lattice D, the canonical envelope in the sense of [42] coincides with the
canonical extension of D. In a somewhat different direction, Coumans [33]
defined canonical extensions for coherent categories and related it to the
topos of types, which is the category of sheaves over a certain frame.

Chapter 2. Duality and canonical extensions for stably compact spaces

In this chapter, we construct a canonical extension for strong proximity lattices in order to give an algebraic, point-free description of a finitary duality for stably compact spaces. In this setting not only morphisms, but also objects may have distinct π- and σ-extensions. This chapter is a modified version of the paper [72].

Strong proximity lattices were introduced, after groundwork of Smyth [135], by Jung and Sünderhauf [88], who showed that these structures are in a dual equivalence with stably compact spaces. Stably compact spaces generalise spectral spaces and are relevant to domain theory in logical form (cf. for example [1], [86] and [100]). In this chapter, we re-examine the Jung-Sünderhauf duality [88] and put it in a broader perspective by connecting it with the theory of canonical extensions.

Outline of the chapter. We first introduce stably compact spaces, motivated by their close relationship with compact ordered spaces, in Section 2.1. In Section 2.2, we set up the categorical preliminaries for duality for stably compact spaces: we relate stably compact spaces to spectral spaces via the Karoubi envelope construction, and we recall the duality between continuous functions on spectral spaces and join-approximable relations on the dual distributive lattices. In Section 2.3, we then discuss proximity lattices, which are algebraic structures that axiomatize the *bases of opens* of stably compact spaces and the relation of way-below-ness, where an open set is *way below* another open set if there is a compact set in between the two. Continuous functions between stably compact spaces correspond to relations between the bases, so that the natural morphisms between proximity lattices are relations, rather than functions. The Karoubi envelope construction is then used to give an abstract categorical proof of a duality between proximity lattices and stably compact spaces. The most important original contribution in this chapter is in Section 2.4, where we introduce and study canonical extensions for proximity lattices with one additional strongness condition. For a proximity lattice that is a basis of open sets for a stably compact space, this strongness condition expresses exactly when an open set is way below the union of a finite collection of open sets. Still in Section 2.4, we start the study of the canonical extensions of morphisms. In Section 2.5, we show how the duality that we obtained via the Karoubi envelope relates to the round spectrum of a proximity lattice [88]. Moreover, we show that,

in the distributive case, canonical extensions of proximity lattices precisely describe their round spectra. We then compare the work in this chapter to the existing literature on this topic and list possible directions for future work.

2.1. Compact ordered and stably compact spaces

In order to motivate the definition of stably compact spaces, we first recall the definition of compact ordered spaces [122].

Definition 2.1.1. A *compact ordered space* is a triple $(X, \tau, \leq)$, where (X, τ) is a compact topological space, and $\leq$ is a partial order on X which is closed as a subset of the product space $(X, \tau) \times (X, \tau)$.

The condition that $\leq$ is closed can be rewritten as a separation property: if $x \nleq y$, then there exist open sets U and V in X such that $x \in U$, $y \in V$, and $(U \times V) \cap \leq = \emptyset$.

Example 2.1.2 (Compact ordered spaces).   1. The real unit interval $[0,1]$ with its usual Euclidean topology and its usual order is a compact ordered space.

   2. Any Priestley space is a compact ordered space. Indeed, a short topological argument shows that Priestley spaces are exactly those compact ordered spaces with the additional property that any open downset is a union of clopen downsets.

   3. Any compact Hausdorff space can be made into a compact ordered space by equipping it with the trivial order, $=$.

Stably compact spaces form the purely topological analogue of compact ordered spaces. As we shall see in Theorem 2.1.10, stably compact spaces can precisely encode the order structure of compact ordered spaces in their topologies. Let us recall the definition of stably compact spaces.

Definition 2.1.3. Let X be a topological space. A set $S \subseteq X$ is called *saturated* if it is an intersection of open sets, and *compact* if any open cover of S contains a finite subcover. The space X is called *locally compact* if, for any open neighbourhood U of a point $x \in X$, there exists an open set V and a compact set K such that $x \in V \subseteq K \subseteq U$. Finally, X is called *stably compact* if X is T_0, sober, locally compact, and the collection of compact-saturated sets is closed under finite intersections.

Example 2.1.4 (Stably compact spaces).   1. The real unit interval $[0,1]$ with the topology $\tau^{\downarrow}$, consisting of those sets which are open in the

Euclidean topology and downsets in the usual order, forms a stably compact space. Moreover, a countable basis for this topology is given by the sets of the form $[0, q)$, where $q \in [0, 1] \cap \mathbb{Q}$. Similarly, the topology $\tau^\uparrow$ of Euclidean-open upsets is stably compact.

2. Any spectral space is a stably compact space. Indeed, spectral spaces are exactly those stably compact spaces in which the compact-open sets form a basis which is closed under finite intersections.

3. Any compact Hausdorff space is a stably compact space.

The reader will have noted the similarity in the lists of Examples 2.1.2 and 2.1.4. This is no coincidence; in fact, the construction of the topology $\tau^\downarrow$ of open downsets (or the topology $\tau^\uparrow$ of open upsets), which was given for the real unit interval $[0, 1]$ in Example 2.1.4.1, can be performed for any compact ordered space and always results in a stably compact space, see Theorem 2.1.10 below.

Definition 2.1.5. Let (X, τ) be a topological space. The *specialization pre-order*[1] on X is defined by

$$x \leq_\tau y \text{ if, and only if, for all open } U, \text{ if } y \in U \text{ then } x \in U.$$

Note that $x \leq_\tau y$ is equivalent to saying that y is in the τ-closure of $\{x\}$. Also, a topological space (X, τ) is T_0 if, and only if, $\leq_\tau$ is anti-symmetric, i.e., a partial order. Finally, we remark that the topological property of being *saturated* (i.e., an intersection of open sets) is equivalent to the order-theoretic property of being *downward closed* in the preorder $\leq_\tau$. In particular, open sets are downsets.

Example 2.1.6 (Specialization orders).   1. Recall that $\tau^\downarrow$ denotes the topology of open downsets on $[0, 1]$. The specialization order of $\tau^\downarrow$ is the usual order of the real unit interval. The specialization order of $[0, 1]$ with the topology of Euclidean-open upsets $\tau^\uparrow$ is the opposite of the usual order of the real unit interval.

2. The specialization order of a spectral space, when viewed as the space of prime ideals of a distributive lattice, is the inclusion order.

3. The specialization order of a compact Hausdorff space is the trivial order, $=$.

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[1] For many authors (e.g. [67], [86]), open sets are upsets rather than downsets, and they therefore define the specialization preorder as the *opposite* of the preorder defined here. In this thesis, we choose to think of open sets as downsets, as this fits well with Stone-Priestley duality for distributive lattices (cf. Section 1.1 above.) This choice ultimately amounts to a matter of taste, which will not be disputed any further at this point.

We see from Example 2.1.6.1 that the *order* of the unit interval $[0, 1]$ can be recovered from the topology of open downsets alone. Can one also recover the usual Euclidean topology on $[0, 1]$? The positive answer to this question relies on the general construction of the *co-compact dual* topology, that we recall now.

Definition 2.1.7. Let (X, ρ) be a topological space. Let ρ^∂ be the topology generated by the complements of ρ-compact-saturated sets. We call the space (X, ρ^∂) the *co-compact dual*[2] of X, and denote it by X^∂.

In a stably compact space, finite unions and arbitrary intersections of compact-saturated sets are still compact-saturated (cf. [86, Lemma 2.8]). Therefore, if (X, ρ) is stably compact, then the collection of complements of compact-saturated sets already forms a topology, which is then by definition the co-compact dual ρ^∂. In fact, more is true:

Theorem 2.1.8 ([86], Theorem 2.12). *Let X be a stably compact space. Then X^∂ is stably compact, and $(X^\partial)^\partial$ is equal to X. In particular, the open sets of X are precisely the complements of compact saturated sets of X^∂.*

Example 2.1.9 (Co-compact duals).   1. The compact-saturated sets in $[0, 1]$ with the topology $\tau^\downarrow$ of Euclidean-open downsets are exactly the downsets that are closed in the Euclidean topology. Thus, the co-compact dual topology of $\tau^\downarrow$ is the topology of Euclidean-open upsets, that is, $(\tau^\downarrow)^\partial = \tau^\uparrow$. Note that the Euclidean topology on $[0, 1]$ is the smallest topology containing both the topology $\tau^\downarrow$ and its co-compact dual.

2. If a spectral space X is the Stone dual space of a distributive lattice D, then the co-compact dual of X is the Stone dual space of the opposite distributive lattice D^{op}.

3. The co-compact dual of a compact Hausdorff space is equal to the space itself.

Mimicking item (1) in the above list of examples, for any space (X, ρ) we can define the *patch topology* ρ^p as the smallest topology containing both ρ and ρ^∂. Conversely, for an ordered topological space $(X, \tau, \leq)$, we let $\tau^\downarrow$ denote the topology of open downsets, and $\tau^\uparrow$ the topology of open upsets. With these constructions in mind, we now recall the following theorem, which seems to be folklore.

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[2]The germ of the idea of the co-compact dual topology appeared in De Groot [74], and it also figures more or less explicitly in Hochster [80]. Some authors thus refer to this topology as the "de Groot dual" or "Hochster dual" topology.

Theorem 2.1.10. *If (X,ρ) is a stably compact space, then $(X,\rho^p,\leq_\rho)$ is a compact ordered space. If $(X,\tau,\leq)$ is a compact ordered space, then $(X,\tau^\downarrow)$ is a stably compact space. Moreover, these two constructions are mutually inverse.*

Proof. See [86, Section 2]. □

Example 2.1.11 (Patch topologies). 1. As we have already seen above, the patch topology of $([0,1],\tau^\downarrow)$ is simply the Euclidean topology on $[0,1]$, and the specialization order is the usual order.

2. If (X,ρ) is the Stone dual space of D, then $(X,\rho^p,\leq)$ is the Priestley dual space of D.

3. If (X,ρ) is a compact Hausdorff space, then $\rho^p = \rho = \rho^\partial$, and $\leq_\rho$ is the trivial order.

Theorem 2.1.10 can be extended to an isomorphism of categories. The natural maps between compact ordered spaces are those which are continuous and order-preserving. The corresponding concept for stably compact spaces is a bit less natural, and a large part of this chapter will in fact be concerned with relaxing the definition of morphism between stably compact spaces.

Definition 2.1.12. A function $f : X \to Y$ between stably compact spaces is called *perfect* if $f^{-1}(U)$ is open for any open set $U \subseteq Y$ (i.e., f is continuous) and $f^{-1}(K)$ is compact for any compact-saturated set $K \subseteq Y$.

Corollary 2.1.13. *The category of compact ordered spaces with continuous order-preserving maps is isomorphic to the category of stably compact spaces with perfect maps.*

Proof. The object assignments of Theorem 2.1.10 can be extended to functors, which act as the identity on morphisms. □

The category of compact ordered spaces admits an involution which sends a compact ordered space $(X,\tau,\leq)$ to $(X,\tau,\geq)$, the same topological space with the opposite order, and is the identity on morphisms. In the isomorphic category of stably compact spaces, this involution is given by the cocompact dual construction: for $(X,\tau,\leq)$ a compact ordered space, we always have $(\tau^\downarrow)^\partial = \tau^\uparrow$.

In this section, we have discussed the correspondence between a class of order-topological spaces and a class of T_0 topological spaces. In the next sections, we will see how these classes can be represented dually as distributive lattices with additional structure. This will in particular lead us to consider the question of representing *continuous*, rather than perfect, maps between stably compact spaces.

2.2. Continuous retracts and the Karoubi envelope

The starting point of lattice representations of stably compact spaces is the following theorem, which relates stably compact spaces to spectral spaces. Recall that a *continuous retraction on a space* X is a continuous function $f : X \to X$ such that $f(f(x)) = f(x)$ for all $x \in X$, or, more concisely, it is an idempotent morphism in the category of topological spaces and continuous functions.

Theorem 2.2.1. *Let Y be a topological space. Then Y is stably compact if and only if there exists a spectral space X and a continuous retraction f on X such that $Y = f(X)$.*

Proof. See [82, Theorem VII.4.6], where a similar statement is proved about stably locally compact locales. Stably compact spaces are sober, so the corresponding fact about locales is in fact equivalent to the present theorem (if we admit the axiom of choice).   □

Using this theorem, we will show (Theorem 2.2.4) that the inclusion of the full subcategory of spectral spaces into the category of stably compact spaces with continuous maps can be described categorically as a *splitting by idempotents* or *Karoubi envelope*, also known as *Cauchy completion* or *idempotent completion*.

Definition 2.2.2. An *idempotent* in a category $\mathcal{C}$ is an endomorphism f such that $f^2 = f$. We say an idempotent $f : X \to X$ *splits* if there are morphisms $r : X \to Y$ and $s : Y \to X$ such that $sr = f$ and $rs = \mathrm{id}_Y$. A category is *Cauchy complete* if all idempotents split. A full and faithful functor $i : \mathcal{C} \to \mathcal{C}^s$ is called a *Karoubi envelope* of $\mathcal{C}$ if it is a universal arrow from $\mathcal{C}$ into a Cauchy complete category. More explicitly, if $\mathcal{C} \to \mathcal{D}$ is a functor and $\mathcal{D}$ is Cauchy complete, then there is an up to natural isomorphism unique factorisation of this functor through the functor i.

Note that two Karoubi envelopes of the same category are equivalent, and that if $\mathcal{C}$ and $\mathcal{C}'$ are equivalent categories, then so are $\mathcal{C}^s$ and $(\mathcal{C}')^s$. Notice also that the category $(\mathcal{C}^s)^{\mathrm{op}}$ is isomorphic to $(\mathcal{C}^{\mathrm{op}})^s$, since 'idempotent' is a self-dual concept. The following lemma will be useful in what follows.

Lemma 2.2.3. *Suppose that $i : \mathcal{C} \to \mathcal{D}$ is a full and faithful functor such that (1) for any idempotent morphism f in $\mathcal{C}$, its image $i(f)$ splits; and (2) any object D of $\mathcal{D}$ is a retract of an object $i(C)$, where $C \in \mathcal{C}$. Then $i : \mathcal{C} \to \mathcal{D}$ is the Karoubi envelope of $\mathcal{C}$.*

Proof. From the two assumptions, one may prove that $\mathcal{D}$ is Cauchy complete. The universal property then follows from standard arguments, see e.g. [16, Proposition 6.5.9].   □

Using Lemma 2.2.3, one may show that any category $\mathcal{C}$ has a Karoubi envelope: it can be constructed as the category $\mathcal{D}$ whose objects are the idempotent morphisms of $\mathcal{C}$ and whose morphisms $g : f \to f'$ are morphisms $g : \text{dom}(f) \to \text{dom}(f')$ in $\mathcal{C}$ such that $f'g = g = gf$. The functor $i : \mathcal{C} \to \mathcal{D}$ sends any object X to id_X and any morphism g in $\mathcal{C}$ to itself, considered as a morphism $\text{id}_{\text{dom}(g)} \to \text{id}_{\text{cod}(g)}$ in $\mathcal{D}$.

Theorem 2.2.4. *The inclusion functor of the full subcategory of spectral spaces into the category of stably compact spaces with continuous maps is a Karoubi envelope.*

Proof. Apply Lemma 2.2.3: condition (1) of the lemma holds by the 'if'-direction of Theorem 2.2.1, and (2) holds by the 'only if'-direction of that theorem. □

In light of Theorem 2.2.4, a stably compact space X can be properly understood as a spectral space 'enriched' with a continuous retraction whose image is the space X. This naturally leads to the idea of dually representing stably compact spaces by distributive lattices that are also enriched with additional structure. To see what additional structure is required, we recall an extension of Stone duality for distributive lattices (cf. Theorem 1.1.7) to the category of spectral spaces with continuous rather than perfect maps (also see, e.g., [2, Section 7.2]).

Definition 2.2.5 ([2], Definition 7.2.24). Let L and M be lattices. A *join-approximable relation* $R : L \twoheadrightarrow M$ is a relation $R \subseteq L \times M$ that satisfies the following conditions, for all $a, a' \in L$, $b, b' \in M$, and finite subsets $A \subseteq L$, $B \subseteq M$:

(a) if $a' \geq aRb \geq b'$, then $a'Rb'$,

(b) if, for all $b \in B$, aRb, then $aR(\bigvee B)$,

(c) if, for all $a \in A$, aRb, then $(\bigwedge A)Rb$,

(d) if $(\bigvee A)Rb$, then there exists finite $B' \subseteq AR(\_)$ such that $b \leq \bigvee B'$.[3]

Lattices with join-approximable relations form a category under relational composition: the relational composition of two join-approximable relations is join-approximable, and, for any lattice L, the reverse order, $\geq_L$, is a join-approximable relation which acts as an identity for relational composition. The following theorem is the *raison d'être* of join-approximable relations.

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[3] Here and in what follows, $AR(\_)$ is shorthand for $\{b \in M \mid \exists a \in L \text{ such that } aRb\}$. Similarly, $(\_)RB$ denotes the set $\{a \in L \mid \exists b \in M \text{ such that } aRb\}$.

Theorem 2.2.6 ([2], Proposition 7.2.5)**.** *The category of spectral spaces and continuous functions is dually equivalent to the category of distributive lattices and join-approximable relations.*

Before we discuss the proof of Theorem 2.2.6, we immediately remark that combining Theorems 2.2.6 and 2.2.4 yields the following (admittedly rather abstract) dual equivalence for stably compact spaces.

Corollary 2.2.7. *The category of stably compact spaces and continuous functions is dually equivalent to the Karoubi envelope of the category of distributive lattices and join-approximable relations.*

In the next section, we will discuss how the Karoubi envelope of the category of distributive lattices and join-approximable relations can be more concretely presented, namely as a category of distributive proximity lattices. We end this section by discussing Definition 2.2.5 and the proof of Theorem 2.2.6.

A continuous function from a spectral space X to a spectral space Y does not need to be perfect, and therefore it does not necessarily correspond, under Stone duality, to a homomorphism from the lattice of compact-open sets of Y to the lattice of compact-open sets of X. However, given such a continuous map $f : X \to Y$ between spectral spaces, the inverse image $f^{-1}(K)$ of a compact-open set K under f is at least open. Therefore, $f^{-1}(K)$ is the union of the compact-open sets in X that are contained in it. To a continuous map $f : X \to Y$, we associate a relation $R_f : E \twoheadrightarrow D$ from the lattice E of compact-open sets of Y to the lattice D of compact-open sets of X, defined by

$$K R_f L \text{ if, and only if, } L \subseteq f^{-1}(K),$$

for any compact-open sets $K \subseteq Y$ and $L \subseteq X$. Note that the assignment $f \mapsto R_f$ is faithful, i.e., different continuous functions yield different dual relations, as can be proved using the facts that Y is T_0, f is continuous, and the compact-open sets form a basis of X. Which relations R between the dual distributive lattices arise in this way? The answer to this question is given in the following proposition, which justifies the definition of join-approximable relation given above.

Proposition 2.2.8. *Let D and E be distributive lattices with Stone dual spaces X and Y, respectively, and let $R : E \twoheadrightarrow D$ be a relation. The following are equivalent:*

1. *There exists a continuous function $f : X \to Y$ such that R is equal to R_f;*

2. *The relation R is a join-approximable relation.*

Proof. This proposition has essentially the same content as [2, Proposition 7.2.25], where it is stated without proof in a point-free setting. We only sketch the proof, as it is mostly routine. For (1) implies (2), it is straightforward to check that if $f : X \to Y$ is continuous, then R_f satisfies (a)–(c) in Definition 2.2.5. We give the proof that R_f satisfies (d), to convey the flavour of the typical argument that is involved here. If $(\bigvee A)R_f b$, then by definition $\widehat{b} \subseteq f^{-1}(\bigcup_{a \in A} \widehat{a})$. For each $x \in \widehat{b}$, pick a_x such that $x \in f^{-1}(\widehat{a_x})$, and then, since $f^{-1}(\widehat{a_x})$ is open, pick $b_x \in D$ such that $x \in \widehat{b_x} \subseteq f^{-1}(\widehat{a_x})$. Now $(\widehat{b_x})_{x \in \widehat{b}}$ covers the compact-open set $\widehat{b}$, so we can pick a finite subcover, which corresponds to a finite $B' \subseteq AR(\_)$ such that $b \leq \bigvee B'$. For (2) implies (1), suppose that R is join-approximable. For $x \in X = D_*$, define its image $f(x)$ to be the point $y \in Y$ corresponding to the prime filter $F_y := (\_)RF_x = \{e \in E \mid \exists d \in F_x : eRd\}$. Indeed, F_y is a filter by (a) and (c), and it is prime by (d). Unravelling the definitions, one shows that for any $e \in E$, $f^{-1}(\widehat{e}) = \bigcup_{eRd} \widehat{d}$, so that f is continuous. From the same equality, compactness of X, and axioms (a) and (b), one deduces that $R = R_f$. □

Proof of Theorem 2.2.6. Write $()^*$ for the functor that assigns the distributive lattice of compact-opens X^* to any spectral space X, and $R_f : Y^* \nrightarrow X^*$ to any continuous function $f : X \to Y$. By Proposition 2.2.8, this functor is well-defined and full to the category of distributive lattices and join-approximable relations, and we already remarked before Proposition 2.2.8 that the functor is faithful. To prove essential surjectivity of $()^*$, first note that isomorphic distributive lattices are also isomorphic in the category of distributive lattices and join-approximable relations. By Stone duality, any distributive lattice is isomorphic to the lattice of compact-open sets of its dual space, and thus also isomorphic to this lattice in the category of distributive lattices with join-approximable relations. □

2.3. Proximity lattices

Jung and Sünderhauf [88] defined "strong proximity lattices" to obtain algebraic structures dual to stably compact spaces. In this section, we will show that the category of join-strong proximity lattices and j-morphisms is the Karoubi envelope of distributive lattices with join-approximable relations (Theorem 2.3.17). It then follows from Corollary 2.2.7 that this category is dually equivalent to the category of stably compact spaces and continuous functions.

Definition 2.3.1 (cf. [88]). A *proximity lattice* is a pair $(L, \prec)$, where L is a lattice and $\prec \subseteq L \times L$ is a relation satisfying the following axioms:

(a) for any $a, b, a', b' \in L$, if $a' \leq a \prec b \leq b'$, then $a' \prec b'$,

(b) for any $b \in L$ and $A \subseteq L$ finite, if, for all $a \in A$, $a \prec b$, then $\bigvee A \prec b$,

(c) for any $a \in L$ and $B \subseteq L$ finite, if, for all $b \in B$, $a \prec b$, then $a \prec \bigwedge B$,

(d) for any $a, b \in L$, $a \prec b$ if, and only if, there exists $c \in L$ such that $a \prec c \prec b$.

We write $\uparrow\! A$ for the set $\{b \in L \mid \exists a \in A : a \prec b\}$ and $\downarrow\! A$ for the set $\{b \in L \mid \exists a \in A : b \prec a\}$. A proximity lattice $(L, \prec)$ is *join-strong* if, for any finite set $B \subseteq L$ and $a \in L$,

(js) if $a \prec \bigvee B$, then there exists a finite set $B' \subseteq \downarrow\! B$ such that $a \prec \bigvee B'$.

Dually, a proximity lattice $(L, \prec)$ is *meet-strong* if, for any finite set $A \subseteq L$ and $b \in L$,

(ms) if $\bigwedge A \prec b$, then there is a finite set $A' \subseteq \uparrow\! A$ such that $\bigwedge A' \prec b$.

A proximity lattice $(L, \prec)$ is *doubly strong* if it is both join- and meet-strong. It is *increasing* if $\prec$ refines the lattice order, i.e., $\prec \subseteq \leq_L$.

It follows directly from this definition that if $(L, \prec)$ is a join-strong proximity lattice, then $(L^{op}, \succ)$ is a meet-strong proximity lattice. The reader will also note the similarity between this definition and the definition of join-approximable relation (Definition 2.2.5). Note, however, that the condition (js) is slightly different from the condition in Definition 2.2.5(d) applied to the relation $\succ$. Jung and Sünderhauf [88] say they choose condition (js) over this more obvious condition, because it allows for an axiomatization that does not mention the lattice order $\leq$ anywhere. The axiom (l_6^*) that was considered by Smyth in [135, p. 330] is equivalent to 2.2.5(d) applied to the relation $\succ$. By the following proposition, for *increasing* join-strong proximity lattices, the two axioms are equivalent. Moreover, as we will see in Proposition 2.3.19, increasing join-strong proximity lattices form a category that is equivalent to the category of all join-strong proximity lattices.

Proposition 2.3.2. *Let L be a lattice and $\prec$ a relation on L.*

1. *If $\succ : L \nrightarrow L$ is join-approximable and idempotent, then $(L, \prec)$ is a join-strong proximity lattice.*

2. *If $(L, \prec)$ is an increasing join-strong proximity lattice, then the relation $\succ : L \nrightarrow L$ is join-approximable and idempotent.*

Proof. Straightforward by comparing Definitions 2.2.5 and 2.3.1.   □

Example 2.3.3. Let X be a stably compact space. Let D be a basis for the open sets which is closed under finite intersections and finite unions. Note that D with the inclusion order is a distributive lattice. Define the "way-below"-relation $\prec$ on D by

$$d \prec e \iff \text{ there exists a compact-saturated set } k \subseteq X \text{ such that } d \subseteq k \subseteq e.$$

We call $(D, \prec)$ an *open-basis presentation* of the space X.
Dually, if E is a 'basis' for the compact saturated sets of X (i.e., every compact-saturated set K of X is an intersection of elements from E) which is closed under finite unions and intersections, we regard it as a distributive lattice with the reverse inclusion order. We then define the relation $\prec$ on E by

$$k \prec l \iff \text{ there exists an open set } u \text{ such that } k \supseteq u \supseteq l,$$

and call $(E, \prec)$ a *compact-saturated-basis presentation* of X.

Proposition 2.3.4.   *1. An open-basis presentation of a stably compact space is a join-strong proximity lattice, which is furthermore increasing and distributive.*

  2. A compact-saturated-basis presentation of a stably compact space is a meet-strong proximity lattice, which is furthermore increasing and distributive.

Proof. In both items, it is not hard to check that all the axioms for a proximity lattice are satisfied. The arguments for join- and meet-strongness are essentially the same as those given in the proof of Theorem 23 in [88]. □

Example 2.3.5. To get a *doubly* strong proximity lattice representing a stably compact space, we can construct a lattice of pairs of open and compact-saturated sets, as was done in section 6 of [88]. We briefly recall this construction.
Let $(D, \prec)$ and $(E, \prec)$ be an open-basis and a compact-saturated-basis presentation of a stably compact space X. Let F be the sublattice of the lattice $D \times E^{\text{op}}$ consisting of those pairs (d, e) for which $d \subseteq e$ as subsets of X. Define the relation $\prec$ on F by $(d, e) \prec (d', e')$ if, and only if, $e \subseteq d'$ as subsets of X.

Proposition 2.3.6 (Theorem 23, [88]). *Let $(F, \prec)$ be defined as in Example 2.3.5. Then $(F, \prec)$ is a doubly strong distributive proximity lattice.*

Example 2.3.7. We consider open-basis presentations for the three examples of stably compact spaces given in Example 2.1.4.

1. An example of an open-basis presentation for the unit interval $[0, 1]$ with the topology $\tau^{\downarrow}$ of Euclidean-open downsets has as its underlying lattice the collection of rational lower open intervals, $D := \{[0, q) \mid q \in \mathbb{Q}\}$. Here, the relation $\prec$ is the strict inclusion order. The collection of all lower open intervals, $D' := \{[0, r) \mid r \in \mathbb{R}\}$, also forms a basis, and $\prec$ is again the strict inclusion order. Note that, although D and D' have different cardinalities, $(D, \prec)$ and $(D', \prec)$ present the same stably compact space.

2. Note that any basis for a space X which is closed under finite unions must contain all compact-open sets of the space. If X is a spectral space, then the lattice D consisting of compact-open sets is a basis. The relation $\prec$ from Example 2.3.3 coincides with the lattice order of D, and $(D, \prec)$ is doubly strong.

3. If X is compact Hausdorff and D is a lattice basis of open sets, then we have $d \prec e$ if, and only if, the closure of d is contained in e.

The fact that any stably compact space X naturally comes with its co-compact dual X^{∂} (Definition 2.1.7) is reflected in the symmetry of join-strong and meet-strong proximity lattices, as the following example explains.

Example 2.3.8. If $(E, \prec_E)$ is a compact-saturated-basis presentation of a stably compact space X, consider the lattice $D := \{X \setminus e \mid e \in E\}$, ordered by inclusion, and define $\prec_D$ on D by $d \prec d'$ iff $(X \setminus d') \prec_E (X \setminus d)$. Then $(D, \prec_D)$ is an open-basis presentation of the co-compact dual space X^{∂}, the lattice D is isomorphic to E^{op}, and $\prec_D = \succ_E$, modulo the lattice isomorphism.

The following observation gives a more algebraic source of examples. If $h : L \twoheadrightarrow M$ is a surjective lattice homomorphism and $\prec_M$ is a proximity relation on M, then one may define a proximity relation $\prec_h$ on L by setting $a \prec_h b$ if, and only if, $h(a) \prec_M h(b)$, that is, $\prec_h := h^{-1}(\prec_M)$. Moreover, $(L, \prec_h)$ will be join- or meet-strong if $\prec_M$ is. Using these observations, one may easily construct doubly strong proximity lattices with a non-increasing proximity relation, as follows.

Example 2.3.9. Let M be a lattice. Consider the homomorphism $h : F_{\mathbf{Lat}}(M) \twoheadrightarrow M$ from the free lattice over the set of generators M to the lattice M, which sends each generator to itself. This homomorphism yields a doubly strong proximity lattice structure $\prec_h$ on $F_{\mathbf{Lat}}(M)$, where $\leq_M$ is regarded as a doubly strong proximity relation on M. The proximity relation $\prec_h$ is not increasing, since the generators of $F_{\mathbf{Lat}}(M)$ form an antichain in $F_{\mathbf{Lat}}(M)$. For a more concrete example, consider the homomorphism h

from the three-element lattice $\mathbf{3} = \{0 \le x \le 1\}$ to $\mathbf{2}$ that sends x to 1. The proximity relation $\prec_h$ on $\mathbf{3}$ is not increasing, since $1 \prec_h x$ but $1 \not\le x$.

Thus, non-increasing proximity relations naturally occur in the context of presentations of a lattice by generators and relations; also see [2, Section 7]. By Example 2.3.7, two proximity lattices may present the same space, even if their underlying lattices are not isomorphic. However, one would like proximity lattices which present the same space to be isomorphic in the category of proximity lattices. Consequently, it should come as no surprise that the notion of morphism for proximity lattices needs to be quite lax.

Definition 2.3.10. Let $(L, \prec_L)$ and $(M, \prec_M)$ be proximity lattices. A relation $R \subseteq L \times M$ is called a *j-morphism* from $(L, \prec_L)$ to $(M, \prec_M)$ if R satisfies (a)–(c) from Definition 2.2.5, and

(d') if $(\bigvee A)Rb$, then there exists $B' \subseteq AR(\_)$ finite such that $b \prec_M \bigvee B'$,

(e) for any $a \in L$, $b \in M$, aRb if, and only if, there exists $c \in L$ such that $a \succ_L cRb$, if, and only if, there exists $d \in M$ such that $aRd \succ_M b$.

By a *proximity morphism* we will mean a relation satisfying (e) and Definition 2.2.5(a)–(c), but not necessarily (d').

Note that Definition 2.3.10(e) says precisely that $\succ_L \circ R = R = R \circ \succ_M$. Condition (d') in this definition is a consequence of (d) in Definition 2.2.5, but it is weaker in general. If the proximity lattice $(M, \prec_M)$ is increasing, then Definition 2.3.10 simplifies:

Proposition 2.3.11. *Let $(L, \prec_L)$ and $(M, \prec_M)$ be proximity lattices, and suppose $\prec_M \subseteq \le_M$. Let $R \subseteq L \times M$ be a relation. The relation R is a j-morphism if, and only if, R is a join-approximable relation from L to M that moreover satisfies $\succ_L \circ R = R = R \circ \succ_M$.*

Join-strong proximity lattices with j-morphisms form a category, **JSPL**, under relational composition: for any join-strong proximity lattice $(L, \prec)$, the identity for the composition is the j-morphism $\succ : (L, \prec) \twoheadrightarrow (L, \prec)$ (cf. [88, Section 7]). Of course, we also have a category **MSPL** of meet-strong proximity lattices with m-morphisms. We now also get two categories of doubly strong proximity lattices, namely the full subcategory **DSPL**$_j$ of **JSPL** and the full subcategory **DSPL**$_m$ of **MSPL**. We will use the terms *j-isomorphic* and *m-isomorphic* to indicate that proximity lattices are isomorphic in the sense of category theory. That is, $(L, \prec_L)$ is j-isomorphic to $(M, \prec_M)$ if there exist j-morphisms $\Phi : (L, \prec_L) \to (M, \prec_M)$ and $\Psi : (M, \prec_M) \to (L, \prec_L)$ such that $\Phi \circ \Psi = \succ_L$ and $\Psi \circ \Phi = \succ_M$. Note that the existence of a j- or m-isomorphism does *not* imply that the underlying lattices L and M are isomorphic.

It should already be clear that the category **JSPL** is close to being the Karoubi envelope of the category of lattices with join-approximable relations. However, before we are able to conclude that these two categories are indeed equivalent (Theorem 2.3.17), we need to study the filter and ideal structure of proximity lattices.

Definition 2.3.12. A non-empty subset $I \subseteq L$ of a proximity lattice $(L, \prec)$ is called a *round ideal* (sometimes $\prec$-ideal), if it is

- $\prec$-*downward closed*: for any $b \in I$, if $a \prec b$, then $a \in I$.

- $\prec$-*up-directed*: for any $a, b \in I$, there is $c \in I$ such that $a \prec c$ and $b \prec c$.

Dually, a *round filter* (or $\prec$-filter), is a subset F of L which is $\prec$-upward closed and $\prec$-down-directed.

Example 2.3.13. Let X be a stably compact space, and $(D, \prec)$ an open-basis presentation of X. Then the round ideals of D, ordered by inclusion, form a frame that is isomorphic to the frame of open sets of X. The isomorphism sends a round ideal I of basic open sets to the open set $U := \bigcup_{d \in I} d$ of X. Dually, if $(E, \prec)$ is a compact-saturated-basis presentation of X then the frame of round filters, ordered by reverse inclusion, is isomorphic to the frame of compact saturated sets of X, ordered by reverse inclusion (which, in turn, is isomorphic to the frame of open sets of the co-compact dual X^∂). The isomorphism sends a round filter F to the compact saturated subset $K := \bigcap_{e \in F} e$ of X. It follows, because X is sober, and $(D, \prec)$ is join-strong, that the space of points of the frame of round ideals is isomorphic to X. The points of this frame correspond precisely to the 'prime round filters' of $(D, \prec)$. We come back to this point in Example 2.5.5 in Section 2.5.

The following alternative characterisation of round ideals and round filters is useful in practice. This characterisation was originally given as the definition in [88].

Lemma 2.3.14. *Let $(L, \prec)$ be a proximity lattice. A subset $I \subseteq L$ is a round ideal if and only if $\downarrow I = I$ and I contains finite joins of its elements. A subset $F \subseteq L$ is a round filter if and only if $\uparrow F = F$ and F contains finite meets of its elements. In particular, round ideals and round filters are always lattice ideals and lattice filters, respectively.*

Proof. The arguments are simple manipulations using the axioms for a proximity lattices and are very similar to those given in section 3 of [88]. □

Since round ideals are closed under arbitrary intersections, the collection $\prec\mathbf{idl}(L)$ of round ideals of a proximity lattice $(L, \prec)$ forms a complete lattice. The same holds for the collection $\prec\mathbf{filt}(L)$ of round filters, ordered by

reverse inclusion. If L is distributive, then $\prec\mathbf{idl}(L)$ and $\prec\mathbf{filt}(L)$ are frames, as proved in [88, Theorem 11]. From the proof of that theorem, we also get the following description of round ideals and filters generated by a set of the form $\downarrow S$.

Proposition 2.3.15. *Let $(L, \prec)$ be a proximity lattice and $S \subseteq L$ a subset. The round ideal generated by $\downarrow S$ is $\{a \in L \mid a \prec \bigvee B \text{ for some finite } B \subseteq S\}$, and the round filter generated by $\uparrow S$ is $\{b \in L \mid \bigwedge A \prec b \text{ for some finite } A \subseteq S\}$.*

Note that this proposition does *not* provide an expression for the round ideal or filter generated by an arbitrary subset S, or even by a single element a, in a proximity lattice. We remark here that the round ideal generated by a single element $a \in L$ can be described as $\{b \in L \mid \uparrow a \subseteq \uparrow b\}$, but we will not need this fact in what follows.
We now characterize proximity morphisms in terms of round ideals.

Lemma 2.3.16. *Let $(L, \prec_L)$ and $(M, \prec_M)$ be proximity lattices, and R a relation from L to M. The following are equivalent:*

1. *The relation R is a proximity morphism,*

2. *For all $a \in L$, $aR(\_)$ is a round ideal, and for all $b \in M$, $(\_)Rb$ is a round filter.*

Furthermore, it follows from these conditions that the map $\prec\mathbf{idl}(R) : I \mapsto IR(\_)$ sends round ideals to round ideals, and that the map $\prec\mathbf{filt}(R) : F \mapsto (\_)RF$ sends round filters to round filters.

Proof. Immediate from the definitions and Lemma 2.3.14. $\square$

Note that a proximity lattice is join-strong if, and only if, the function $\downarrow : L \to \prec\mathbf{idl}(L)$ preserves finite joins ([88, Proposition 17]). Similarly, one can show using Proposition 2.3.15 that, if $R : (L, \prec_L) \twoheadrightarrow (M, \prec_M)$ is a proximity morphism, then R is a j-morphism if, and only if, the associated map $L \to \prec\mathbf{idl}(M)$, which sends $a \in L$ to $aR(\_)$, preserves finite joins.

We are now ready to characterize the category of join-strong proximity lattices compared to the category of lattices with join-approximable relations.

Theorem 2.3.17. *The category of join-strong (distributive) proximity lattices with j-morphisms is equivalent to the Karoubi envelope of the category of (distributive) lattices with join-approximable relations.*

Proof. We will give the proof for the category of all lattices; the proof for the category of distributive lattices is the same. Let i be the functor from lattices to join-strong proximity lattices which sends a lattice L to $i(L) := (L, \leq)$

and a join-approximable relation R to the j-morphism $R : (L, \leq) \twoheadrightarrow (L, \leq)$. We show that i satisfies the two conditions in Lemma 2.2.3. To see that (1) holds, let $P : L \twoheadrightarrow L$ be an idempotent join-approximable relation. Now $(L, \prec_P)$, where $\prec_P$ is the relation P^{-1}, is a join-strong proximity lattice by Proposition 2.3.2.1, and $\mathrm{id}_{(L, \prec_P)} = P$. Therefore, $i(P) = P : (L, \leq) \twoheadrightarrow (L, \leq)$ splits as the composition of the j-morphisms $P : (L, \leq) \twoheadrightarrow (L, \prec_P)$ and $P : (L, \prec_P) \twoheadrightarrow (L, \leq)$. To see that (2) holds, we show that any join-strong proximity lattice $(L, \prec)$ is a retract of the lattice $(\prec\mathbf{idl}(L), \leq)$, by defining j-morphisms R and S such that $S \circ R$ is the identity on $(L, \prec)$. The j-morphism $R : (\prec\mathbf{idl}(L), \leq) \twoheadrightarrow (L, \prec)$ is defined by IRa if, and only if, $a \in I$. The j-morphism $S : (L, \prec) \twoheadrightarrow (\prec\mathbf{idl}(L), \leq)$ is defined by aSI if, and only if, there exists $b \prec a$ such that $I \subseteq {\downarrow}b$. It is easy to verify that $S \circ R = {\succ} = \mathrm{id}_{(L, \prec)}$. $\square$

Composing the equivalence of this theorem with the dual equivalence in Corollary 2.2.7, we now have a categorical proof of the following duality theorem for stably compact spaces.

Corollary 2.3.18. *The category of join-strong distributive proximity lattices with j-morphisms is dually equivalent to the category of stably compact spaces with continuous functions.*

In Section 2.5, we will discuss the concrete content of this theorem in some more detail. Jung and Sünderhauf [88] stress that it is not necessary to assume that the relation $\prec$ of a proximity lattice $(L, \prec)$ is increasing (i.e., contained in the lattice order $\leq$). However, making this assumption does not change the category, up to equivalence:

Proposition 2.3.19. *Every join-strong proximity lattice $(L, \prec)$ is j-isomorphic to the increasing join-strong proximity lattice $(\prec\mathbf{idl}(L), \ll)$, where $\ll$ is the way-below relation in the complete lattice of round ideals of L.*

Proof. One may calculate that the way-below relation on $\prec\mathbf{idl}(L)$ says, for round ideals I and J, that $I \ll J$ iff there exists $d \in J$ such that $I \subseteq {\downarrow}d$. The j-isomorphism is given by the j-morphisms $\Phi : (L, \prec) \to (\prec\mathbf{idl}(L), \ll)$ and $\Psi : (\prec\mathbf{idl}(L), \ll) \to (L, \prec)$ defined by $a \Phi I$ iff $I \ll {\downarrow}a$ and $I \Psi a$ iff $a \in I$. It is not hard, but a bit tedious, to check that Φ and Ψ are indeed j-morphisms. To conclude, note that $\Phi \circ \Psi = {\succ}$ and $\Psi \circ \Phi = {\gg}$. $\square$

Now, for increasing proximity lattices, we have the following fact.

Proposition 2.3.20. *Let $(L, \prec)$ be an increasing join-strong proximity lattice. The relation $\prec$ is reflexive if, and only if, $\prec$ is equal to the lattice order $\leq_L$ of L.*

Proof. The 'if' direction is clear. For 'only if', note that we already have $\prec \subseteq \leq_L$ since $(L, \prec)$ is assumed to be increasing. For the inclusion $\leq_L \subseteq \prec$,

suppose $a \leq_L b$. Then $a \wedge b = a$, and since $\prec$ is reflexive we have $a \prec a$, so $a \prec a \wedge b$. From the proximity axiom for $\wedge$, we conclude that $a \prec b$. $\qquad \square$

We will come back to the property of reflexivity and how it can make the theory of proximity lattices collapse in Proposition 2.4.9, after we introduce the canonical extension in the next section.

2.4. Canonical extensions of proximity lattices

In this section we show that canonical extensions can be generalized to proximity lattices. We present the material in this section without the assumption that the proximity lattices involved are distributive, analogously to the work on canonical extension for lattices in [54]. Later, in Theorem 2.5.7, we will show that in the case of a *distributive* join-strong proximity lattice, the canonical extension is exactly the lattice of saturated sets of its dual space.

For a proximity lattice version of canonical extension, we parametrize the definition of canonical extension (Definition 1.2.2) in the proximity relation $\prec$, as follows.

Definition 2.4.1. Let $(L, \prec)$ be a proximity lattice and $h : L \to C$ a lattice homomorphism with C a complete lattice. We call $u \in C$ a *round-ideal element* if there is a round ideal I of L such that $u = \bigvee h[I]$. Dually, u is a *round-filter element* if there is a round filter F of L such that $u = \bigwedge h(F)$. We denote the set of round-ideal elements of the extension h by $I^h_\prec(C)$, and the set of round-filter elements by $F^h_\prec(C)$ (usually, when the map h is fixed, we just write $I_\prec(C)$ and $F_\prec(C)$). We say a function $h : L \to C$ is a *π-canonical extension* of the proximity lattice $(L, \prec)$ if, for all $u, v \in C$, $S, T \subseteq L$, and $a \in L$:

1. (round-dense) if $u \nleq v$, then there exist a round-filter element x and a round-ideal element y such that $x \leq u$, $v \leq y$, and $x \nleq y$ in C.

2. (round-compact) if $\bigwedge h(\uparrow S) \leq \bigvee h(\downarrow T)$ in C, then there exist finite sets $S' \subseteq S$ and $T' \subseteq T$ such that $\bigwedge S' \prec \bigvee T'$.

3. (round-join-preserving) For all $a \in L$, $h(a) = \bigvee\{h(b) \mid b \prec a\}$.

Dually, $k : L \to C$ is a *σ-canonical extension* of the proximity lattice $(L, \prec)$ if it is round-dense, round-compact, and round-meet-preserving:

3'. (round-meet-preserving) For all $a \in L$, $k(a) = \bigwedge\{k(b) \mid b \succ a\}$.

Note that if an extension h is round-join-preserving or round-meet-preserving, it follows in both cases that h is $\prec$-*preserving*, i.e., for all $a, b \in L$, if $a \prec b$, then $h(a) \leq h(b)$. Note that the maps h and k are not necessarily injective: it is not hard to check that the map $h : \mathbf{3} \to \mathbf{2}$ from Example 2.3.9 is both a π- and σ-canonical extension.

Before showing existence and uniqueness of the canonical extensions in the presence of strongness axioms, we now give some useful alternative characterisations of round-denseness and round-compactness. The reader may recognize these as the proximity-lattice versions of usual lattice-theoretical facts, and the proofs are straight-forward generalisations of these proofs.

Proposition 2.4.2. *Let* $(L, \prec)$ *be a proximity lattice. The following are equivalent for any* $\prec$-*preserving extension* $h : L \to C$.

1. *The extension h is round-compact,*

2. *For every round filter F and round ideal I of L such that* $\bigwedge h(F) \leq \bigvee h(I)$ *in C, we have* $F \cap I \neq \varnothing$.

Proof. For the direction (1) $\Rightarrow$ (2), the definition of round-compactness gives finite subsets S' of F and T' of I such that $\bigwedge S' \prec \bigvee T'$. We then have $\bigvee T' \in I$ because I is an ideal. Also, $\bigwedge S' \in F$ since F is a filter, and then, since F is round, we also have $\bigvee T' \in F$. Hence $\bigvee T' \in F \cap I$. For the direction (2) $\Rightarrow$ (1), let S and T be subsets of L such that $\bigwedge h(\uparrow S) \leq \bigvee h(\downarrow T)$. Let F be the round filter generated by $\uparrow S$ and I the round ideal generated by $\downarrow T$. Then $\bigwedge h(F) \leq \bigwedge h(\uparrow S) \leq \bigvee h(\downarrow T) \leq \bigvee h(I)$. By (2), pick $a \in F \cap I$. By Proposition 2.3.15, there exist finite subsets $S' \subseteq S$ and $T' \subseteq T$ such that $\bigwedge S' \prec a \prec \bigvee T'$. Hence, $\bigwedge S' \prec \bigvee T'$, as required. $\square$

Proposition 2.4.3. *The following are equivalent for any extension* $h : L \to C$.

1. *The extension h is round-dense,*

2. *For any* $u \in C$, $u = \bigvee \{x \mid u \geq x \in F^h_\prec(C)\}$ *and*
   $u = \bigwedge \{y \mid u \leq y \in I^h_\prec(C)\}$.

Proof. This is a simple rewriting of the definition of round-dense. $\square$

We will now present the π-canonical extension of a join-strong proximity lattice as a lattice of Galois-closed sets, and show that it is unique up to isomorphism. For this, we first recall some elementary facts about polarities and Galois connections that we will need. We refer the reader to [50] and [55] for more details.

A *polarity* is a triple (X, Y, Z) where X and Y are sets and $Z \subseteq X \times Y$. Any polarity gives rise to a pair of functions (l_Z, r_Z) between the posets $\mathcal{P}(X)$

and $\mathcal{P}(Y)$, where the function $l_Z : \mathcal{P}(X) \to \mathcal{P}(Y)$ sends a subset $u \subseteq X$ to $\{y \mid \forall x \in u : xZy\}$ and $r_Z : \mathcal{P}(Y) \to \mathcal{P}(X)$ sends a subset $v \subseteq Y$ to $\{x \mid \forall y \in v : xZy\}$. This pair of functions forms a Galois connection, i.e., an adjunction between $\mathcal{P}(X)^{\mathrm{op}}$ and $\mathcal{P}(Y)$, since $v \subseteq l_Z(u)$ if, and only if, $u \subseteq r_Z(v)$. The composition $c_Z := r_Z \circ l_Z$ is therefore a closure operator on $\mathcal{P}(X)$, and we denote by $C := \mathcal{G}(X, Y, Z)$ the complete lattice of closed sets, i.e., $u \subseteq X$ such that $c_Z(u) = u$. We have maps $f : X \to C$ and $g : Y \to C$ which are given by $x \mapsto c_Z(\{x\})$ and $y \mapsto r_Z(\{y\})$, respectively.

Theorem 2.4.4. *Let (X, Y, Z) be a polarity.*

1. *The complete lattice $C := \mathcal{G}(X, Y, Z)$ has the following properties.*

    (a) *For any $u \in C$, $u = \bigvee \{f(x) \mid x \in X, \ f(x) \leq u\}$,*

    (b) *For any $u \in C$, $u = \bigwedge \{g(y) \mid y \in Y, \ u \leq g(y)\}$,*

    (c) *For any $x \in X$, $y \in Y$, we have $f(x) \leq g(y)$ iff xZy.*
    Moreover, it follows from (a)–(c) that

    (d) *For $x_1, x_2 \in X$, $f(x_1) \leq f(x_2)$ if, and only if, for all $y \in Y$, $x_2 Zy$ implies $x_1 Zy$,*

    (e) *For $y_1, y_2 \in Y$, $g(y_1) \leq g(y_2)$ if, and only if, for all $x \in X$, xZy_1 implies xZy_2,*

    (f) *For $y \in Y$, $x \in X$, $g(y) \leq f(x)$ if, and only if, for all $x' \in X, y' \in Y$, if $x'Zy$ and xZy' then $x'Zy'$.*

2. *If C' is a complete lattice and $f' : X \to C'$, $g' : Y \to C'$ are functions such that properties (a)–(c) from (1) also hold for C', f' and g', then there is a unique complete lattice isomorphism $\varphi : C' \to C$ such that $\varphi \circ f' = f$ and $\varphi \circ g' = g$.*

3. *Let $Q = (X \sqcup Y, \preccurlyeq)$ be the pre-order defined by items (c)–(f) of (1). Then the Dedekind-MacNeille completion C'' of Q, together with the natural inclusion maps of $f'' : X \to C''$ and $g'' : Y \to C''$, satisfies (a)–(c) of item (1), and hence, in particular, it is uniquely isomorphic to C.*

Proof. See, for example, [50, Section 2]. □

In order to construct the π-canonical extension of a join-strong proximity lattice, we now associate the following polarity (X, Y, Z) to a proximity lattice $(L, \prec)$:

$$X := \prec\mathbf{filt}(L), \ \ Y := \prec\mathbf{idl}(L), \ \ Z := \{(F, I) \mid F \cap I \neq \varnothing\}. \quad (2.1)$$

Let $C := \mathcal{G}(X, Y, Z)$ be the associated complete lattice, and let $h : L \to C$ be the function given by $h(a) := g(\downarrow a)$. We will now show in a few steps

that $h : L \to C$ is indeed a π-canonical extension of $(L, \prec)$. Note first that $h(a) = \{F : F \cap \downarrow a \neq \varnothing\} = \{F : a \in F\}$, and hence in particular that h is $\prec$-preserving.

Lemma 2.4.5. *If $(L, \prec)$ is a join-strong proximity lattice, then $h : L \to C$ defined from the polarity in (2.1) is a homomorphism.*

Proof. Since $a \wedge b \in F$ iff $a \in F$ and $b \in F$, it is clear that h preserves binary meets. Also, $h(\top_L) = \top_C$ because any round filter contains $\top_L$, and $h(\bot_L) = c_Z(\varnothing) = \bot_C$. Since h preserves meets, it preserves order; it remains to show that $h(a \vee b) \leq h(a) \vee h(b)$. To this end, let $F \in h(a \vee b)$, which means that $a \vee b \in F$. We need to show that $F \in h(a) \vee h(b) = c_Z(h(a) \cup h(b))$. Let $I \in l_Z(h(a) \cup h(b))$ be arbitrary. Since F is round, pick $x \in F$ such that $x \prec a \vee b$. By join-strongness, pick $a', b' \in L$ such that $a' \prec a$, $b' \prec b$, and $x \prec a' \vee b'$. Since $x \in F$, we get $a' \vee b' \in F$. On the other hand, $a \in \uparrow a'$, so, since $I \in l_Z(h(a))$, we get $\uparrow a' \cap I \neq \varnothing$. It follows that $a' \in I$, and similarly that $b' \in I$. We conclude that $a' \vee b' \in I$, so $F \cap I \neq \varnothing$, as we needed to show. $\qquad\square$

The following lemma identifies the round-filter and round-ideal elements of the extension h. This lemma could alternatively be obtained as a consequence of a more general fact about the construction in Theorem 2.4.4, in the case where the sets X and Y in the polarity themselves have additional lattice structure, as is indeed the case in (2.1).

Lemma 2.4.6. *Let $h : L \to C$ be the extension defined from the polarity (X, Y, Z) in (2.1). The set of round-filter elements of h is exactly $f(X)$, and the set of round-ideal elements of h is exactly $g(Y)$.*

Proof. We show that, for any round filter F, we have $f(F) = \bigwedge h(F)$, which clearly suffices to conclude the first part. Let F be a round filter. If $a \in F$, then $F Z \downarrow a$, so $f(F) \leq g(\downarrow a) = h(a)$. We conclude that $f(F) \leq \bigwedge h(F)$. For the other inequality, write $u := \bigwedge h[F]$. Then, using Theorem 2.4.4.1(a), $u = \bigvee \{f(F') \mid f(F') \leq u\}$. Let F' be arbitrary with $f(F') \leq u$. Then $f(F') \leq h(a)$ for all $a \in F$, so $F \subseteq F'$. Therefore, any round ideal I which intersects F intersects F', in other words, $f(F') \leq f(F)$, by 2.4.4.1(c). The proof of the second part is dual. $\qquad\square$

Proposition 2.4.7. *Let $(L, \prec)$ be a join-strong proximity lattice. The function $h : L \to C$, defined with the polarity (X, Y, Z) above, is a π-canonical extension of L.*

Proof. We showed that h is a homomorphism in Lemma 2.4.5. To see that h is round-dense, recall from Theorem 2.4.4.1(a,b) that the set $f(X)$ join-generates C and the set $g(Y)$ meet-generates C. Round-denseness follows

by Proposition 2.4.3 and Lemma 2.4.6. For round-compactness, 2.4.4.1(c) yields that for any polarity (X, Y, Z), if $x \in X$ and $y \in Y$, then $x \leq y$ in $\mathcal{G}(X, Y, Z)$ if and only if xZy. In particular, in our case, if F is a round filter and I is a round ideal, then $\bigwedge F \leq \bigvee I$ in $\mathcal{G}(X, Y, Z)$ implies that $F \cap I \neq \emptyset$, which, by Proposition 2.4.2, is equivalent to round-compactness. Finally, to prove that h is round-join-preserving, let $a \in L$ be arbitrary. We need to show that $h(a) = \bigvee \{h(b) \mid b \prec a\}$. We already noted before Lemma 2.4.5 that h is $\prec$-preserving; therefore, $h(a) \geq \bigvee \{h(b) \mid b \prec a\}$. For the converse inequality, if $F \in h(a)$, that is, $a \in F$, then since F is round, there is some $b \in F$ such that $b \prec a$. We conclude that $F \in h(b)$, which is below the join. $\qquad \square$

We now also prove uniqueness of the π-extension $h : L \to C$ constructed above.

Proposition 2.4.8. *If $(L, \prec)$ is a join-strong proximity lattice and $h' : L \to C'$ is a π-canonical extension of L, then there exists a complete lattice isomorphism $\varphi : C' \to C$ such that $\varphi \circ h' = h$.*

Proof. The homomorphism h' induces a function $f' : X \to C'$, defined by $f'(F) := \bigwedge_{a \in F} h'(a)$ and $g' : Y \to C'$ defined by $g'(I) := \bigvee_{a \in I} h'(a)$. It follows from round-denseness that $f'[X]$ join-generates C' and $g'[Y]$ meet-generates C'. It follows from round-compactness of h' that $f'(F) \leq g'(I)$ implies $F \cap I \neq \emptyset$, and the other implication holds by the definition of f' and g'. Therefore, by Theorem 2.4.4.2, there is a unique isomorphism $\varphi : C' \to C$ such that $\varphi \circ f' = f$ and $\varphi \circ g' = g$. Since h' is round-join-preserving, we have $h'(a) = \bigvee \{h'(b) \mid b \prec a\} = g'(\downarrow a)$, so we deduce from $\varphi \circ g' = g$ that $\varphi \circ h' = h$. $\qquad \square$

The same existence and uniqueness results hold for meet-strong proximity lattices and their σ-extensions. In order to prove that a meet-strong proximity lattice $(L, \prec)$ has a σ-extension, consider the join-strong proximity lattice $(L^{\mathrm{op}}, \succ)$. Let $h : L^{\mathrm{op}} \to C$ be a π-extension of $(L^{\mathrm{op}}, \succ)$. Then $k : L \to C^{\mathrm{op}}$, defined by $k(a) := h(a)$, is a σ-extension of $(L, \prec)$. More explicitly, given a meet-strong proximity lattice $(L, \prec)$, one may consider the polarity $(\prec\mathbf{idl}(L), \prec\mathbf{filt}(L), Z^{-1})$, where the relation Z is defined as before. Define the function $k : M \to \mathcal{G}(\prec\mathbf{idl}(L), \prec\mathbf{filt}(L), Z)^{\mathrm{op}}$ by $a \mapsto g(\uparrow a)$. Then, to be able to show that k is a homomorphism, one needs to assume that $(L, \prec)$ is meet-strong: the situation is order-dual to that in the proof of Lemma 2.4.5. The rest of the proof that k is a σ-extension is analogous to the proof of Proposition 2.4.7. The canonical extensions of a proximity lattice $(L, \prec)$, if they exist, will be denoted by $h : (L, \prec) \to (L, \prec)^{\pi}$ and $k : (L, \prec) \to (L, \prec)^{\sigma}$.

If a proximity lattice $(L, \prec)$ is doubly strong, then both canonical extensions exist. Note that the complete lattices $\mathcal{G}(X, Y, Z)$ and $\mathcal{G}(Y, X, Z^{-1})^{\mathrm{op}}$ which were used to define the π- and σ-extension, respectively, are isomorphic. However, the maps giving the extension, i.e., $h : L \to \mathcal{G}(X, Y, Z)$ and $k : L \to \mathcal{G}(Y, X, Z^{-1})^{\mathrm{op}}$, are not always the same. We have an easy characterization of when they do coincide.

Proposition 2.4.9. *If $(L, \prec)$ is a doubly strong proximity lattice, then the following are equivalent:*

1. *The σ-extension $k : (L, \prec) \to (L, \prec)^\sigma$ is also a π-extension of $(L, \prec)$,*

2. *There exists an isomorphism $\varphi : (L, \prec)^\pi \to (L, \prec)^\sigma$ such that $\varphi \circ h = k$,*

3. *The relation $\prec$ is reflexive.*

Proof. That (1) implies (2) follows directly from the uniqueness of the π-extension (Proposition 2.4.8). For (2) implies (3), note that because $(L, \prec)^\sigma$ is now also a π-extension, we get, in the concrete representation of $(L, \prec)^\sigma$ as $\mathcal{G}(X, Y, Z)$, for any $a \in L$, that $h(a) = g(\downarrow a) = \bigwedge\{g(\downarrow b) : a \prec b\}$. Observe that the round filter $\uparrow a$ is an element of the right-hand-side. Hence, $\uparrow a$ is in $g(\downarrow a)$, which implies by round-compactness that $a \in \downarrow a$. For (3) implies (1), note that the requirements of round-join-preserving and round-meet-preserving both become equivalent to $\prec$-preservation in the case where $\prec$ is reflexive. $\qquad\square$

Recall from Proposition 2.3.20 that for increasing proximity lattices, the relation $\prec$ is reflexive if, and only if, it is equal to the lattice order. In particular, if L is a lattice, then the π- and σ-extensions of $(L, \leq)$, viewed as a doubly strong proximity lattice, coincide with the usual canonical extension of the lattice L. We will come back to the relation between the two canonical extensions of a doubly strong proximity lattice in Proposition 2.5.8, after discussing the dualities for proximity lattices. In the distributive case, we will see that R is reflexive if and only if $(L, \prec)$ is j-isomorphic to a proximity lattice of the form $(M, \leq_M)$, where $\leq_M$ is the lattice order on M.

We fix the following notation for the rest of this section: $R : (L, \prec) \twoheadrightarrow (M, \prec)$ is a proximity morphism between join-strong proximity lattices and $h_L : (L, \prec) \to (L, \prec)^\pi$ and $h_M : (M, \prec) \to (M, \prec)^\pi$ are the π-canonical extensions of $(L, \prec)$ and $(M, \prec)$. Additional assumptions on R, where needed, will be mentioned in the statements of the results. We now define the π-extension of R, a map from $(L, \prec)^\pi$ to $(M, \prec)^\pi$ which *extends R*, in a sense to be made precise below. Recall from Lemma 2.3.16 that, for any round ideal I, the set $IR(\_)$ is a round ideal. Now, a round-ideal element

$y \in I_\prec((L, \prec)^\pi)$ represents the round ideal $I := h_L^{-1}(\downarrow y)$. So R^π should map y to the ideal element in $(M, \prec)^\pi$ that represents the round ideal $IR(\_)$. Since the round-ideal elements meet-generate the lattice $(L, \prec)^\pi$, we now simply extend the assignment by taking meets. The formal definition is as follows.

Definition 2.4.10. Let $R^\pi : I_\prec((L, \prec)^\pi) \to I_\prec((M, \prec)^\pi)$ be the function defined, for y a round-ideal element of $(L, \prec)^\pi$, by

$$R^\pi(y) := \bigvee\{h_M(b) \mid b \in M \text{ s.t. } \exists a \in L : aRb \text{ and } h_L(a) \leq y\}.$$

Now let $R^\pi : (L, \prec)^\pi \to (M, \prec)^\pi$ be the function defined by

$$R^\pi(u) := \bigwedge\{R^\pi(y) : u \leq y \in I_\prec((L, \prec)^\pi)\}.$$

Dually, we could define the σ-extension of an m-morphism between meet-strong proximity lattices.

It is immediate from the definition that R^π is order-preserving. We now show in which sense the function R^π extends the proximity morphism R.

Lemma 2.4.11. *For any $a \in L$, we have*

$$R^\pi(h_L(a)) = \bigvee\{h_M(b) \mid aRb\}.$$

Proof. Note that $h_L(a)$ is a round-ideal element. Therefore, by definition, we have $R^\pi(h_L(a)) = \bigvee\{h_M(b) \mid b \in M \text{ s.t. } \exists a' \in L : a'Rb \text{ and } h_L(a') \leq h_L(a)\}$. Hence, it is clear that if aRb then $h_M(b) \leq R^\pi(h_L(a))$. Therefore, we have $R^\pi(h_L(a)) \geq \bigvee\{h_M(b) \mid aRb\}$. For the converse inequality, let $b \in M$ and $a' \in L$ such that $a'Rb$ and $h_L(a') \leq h_L(a)$. We have $\bigwedge h_L((\_)Rb) \leq h_L(a')$, which is in turn below $h_L(a) = \bigvee h_L(\downarrow a)$. By Proposition 2.4.2, since $(\_)Rb$ is a round filter and $\downarrow a$ is a round ideal, there exists $a'' \prec a$ with $a''Rb$. Since R is a proximity morphism, we conclude that aRb. □

We now discuss the meet-preservation of R^π.

Lemma 2.4.12. *The function R^π preserves all meets of collections of round-ideal elements.*

Proof. Let U be a collection of round-ideal elements of $(L, \prec)^\pi$. Writing $u_0 := \bigwedge U$, we need to show that $R^\pi(u_0) = \bigwedge\{R^\pi(u) \mid u \in U\}$. Since R^π is order-preserving, it suffices to show that $R^\pi(u_0) \geq \bigwedge\{R^\pi(u) \mid u \in U\}$. For this, we use round-denseness. Let F be an arbitrary round filter such that $x_0 := \bigwedge h_M(F) \leq \bigwedge\{R^\pi(u) \mid u \in U\}$. We show that $x_0 \leq R^\pi(u_0)$. For any $u \in U$, we have that $x_0 \leq R^\pi(u)$. By round-compactness (Proposition 2.4.2) and the definition of R^π for round-ideal elements, pick $b_u \in F$ and $a'_u \in L$

such that $h_L(a'_u) \leq y$ and $a'_u R b_u$. Since $\succ \circ R = R$, we can pick $a_u \in L$ such that $a_u \prec a'_u$ and $a_u R b_u$. Performing these steps for every $u \in U$, we get subsets $\{a_u : u \in U\}$ of L and $\{b_u : u \in U\}$ of F. Let G be the round filter generated by $\uparrow\{a_u : u \in U\}$, and let $v := \bigwedge h_L(G)$. Now, since for every $u \in U$ we have $a'_u \in G$, we get $v \leq h_L(a'_u) \leq u$, so that $v \leq \bigwedge U = u_0$, and hence $R^\pi(v) \leq R^\pi(u_0)$. We finish the argument by showing that $x_0 \leq R^\pi(v)$. By definition of R^π, we may show that for an arbitrary round ideal I of L such that $y_0 := \bigvee h_L(I) \geq v$, we have $x_0 \leq R^\pi(y_0)$. By round-compactness, if $v \leq \bigvee h_L(I)$, then there is $a_0 \in G \cap I$. Since $a \in G$, by Proposition 2.3.15, there are $u_1, \ldots, u_n \in U$ such that $\bigwedge_{i=1}^n a_{u_i} \prec a_0$. Since $a_{u_i} R b_{y_i}$ for every i, we have that $\bigwedge_{i=1}^n a_{u_i} R \bigwedge_{i=1}^n b_{u_i}$. We now put $b_0 := \bigwedge_{i=1}^n b_{u_i}$, which is in F since all the b_{u_i} are, and we get that $a_0 R b_0$ since $\succ \circ R = R$. Since $a_0 \in I$, we have $h_L(a_0) \leq y_0$, and so, since $a_0 R b_0$, we have $h_M(b_0) \leq R^\pi(y_0)$, by the definition of $R^\pi(y_0)$. Since $x_0 = \bigwedge h_M(F)$, and $b_0 \in F$, we get $x_0 \leq h_M(b_0)$, so we conclude $x_0 \leq R^\pi(y_0)$. □

Using this lemma, it is now fairly easy to show the following.

Proposition 2.4.13. *The function R^π preserves all meets.*

Proof. Let U be an arbitrary collection of elements from $(L, \prec)^\pi$. We define U' to be the set of round-ideal elements below U, i.e., $U' := I_\prec((L, \prec)^\pi) \cap \downarrow U$. Note that $\bigwedge U' \leq \bigwedge U$, by round-denseness. We further have that $\bigwedge R^\pi(U)$ is a lower bound for the set $R^\pi(U')$, so $\bigwedge R^\pi(U) \leq \bigwedge R^\pi(U')$. Finally, since U' is a set of round-ideal elements, we have by Lemma 2.4.12 that $\bigwedge R^\pi(U') \leq R^\pi(\bigwedge U')$. Putting these inequalities together, we get

$$\bigwedge R^\pi(U) \leq \bigwedge R^\pi(U') \leq R^\pi\left(\bigwedge U'\right) \leq R^\pi\left(\bigwedge U\right). \qquad \square$$

Regarding joins, the situation is more delicate. However, we can show the following, by similar methods to the ones above.

Lemma 2.4.14. *The function R^π preserves joins of up-directed collections of round-ideal elements.*

Proof. Let U be an up-directed collection in $I_\prec((L, \prec)^\pi)$. We need to show that $R^\pi(\bigvee U) \leq \bigvee R^\pi(U)$, as the other direction follows directly from the fact that R^π is order-preserving. Note that $\bigvee U = \bigvee h_L(I)$, where we write I for the round ideal $\downarrow(\bigcup_{u \in U} h_L^{-1}(\downarrow u))$ of L, so $\bigvee U$ is a round-ideal element. Therefore, by definition of R^π we have that $R^\pi(\bigvee U)$ is equal to $\bigvee\{h_M(b) \mid \exists a \in L : h_L(a) \leq \bigvee U \text{ and } a R b\}$. Let $a \in L$ and $b \in M$ such that $h_L(a) \leq \bigvee U$ and $a R b$. We need to show $h_M(b) \leq \bigvee R^\pi(U)$. Pick a' such that $a' R b$ and $a' \prec a$. Then

$$\bigwedge h_L(\uparrow a') \leq h_L(a) \leq \bigvee U = \bigvee h_L(I).$$

By round-compactness, pick $t_1, \ldots, t_n \in I$ such that $a' \prec \bigvee_{i=1}^n t_i$. Since the extension h_L is $\prec$-preserving, for each i, there exists $u_i \in U$ such that $h_L(t_i) \le u_i$. Since U is up-directed, pick $u_0 \in U$ such that $u_0 \ge \bigvee_{i=1}^n u_i$. We then have $h_L(\bigvee_{i=1}^n t_i) \le \bigvee_{i=1}^n u_i \le u_0$. Now $a' \prec \bigvee_{i=1}^n t_i$ and $a'Rb$, so $(\bigvee_{i=1}^n t_i)Rb$. We conclude, by definition of $R^\pi(u_0)$, that $h_M(b) \le R^\pi(u_0)$, which is below $\bigvee R^\pi(U)$, as required. $\qquad\square$

If R is a j-morphism, then we can also show the following.

Lemma 2.4.15. *If $R : (L, \prec) \to (M, \prec)$ is a j-morphism, then R^π preserves finite joins of round-ideal elements.*

Proof. Suppose $U = \{u_1, \ldots, u_n\}$ is a finite subset of $I_\prec((L, \prec)^\pi)$. We need to show that $R^\pi(\bigvee U) \le \bigvee R^\pi(U)$. For each $1 \le k \le n$, let I_k be a round ideal such that $u_k = \bigvee h_L(U_k)$. By the same argument as in the proof of Lemma 2.4.14, $\bigvee U$ is a round-ideal element. Take an arbitrary $a \in L$ and $b \in M$ such that $h_L(a) \le \bigvee U$ and aRb. We show that $h_M(b) \le \bigvee R^\pi(U)$. By denseness, it suffices to show that for an arbitrary round-filter element $x \le h_M(b)$, we have $x \le \bigvee R^\pi(U)$. Since x is a round-filter element, there is a round filter F such that $x = \bigwedge h_M[F]$, and then by round-compactness and round-join-preservingness there is some $c \in F \cap {\downarrow}b$. Now $G := (\_)RF$ is a round filter by Lemma 2.3.16. Moreover, $a \in G$ since $aRb \succ c$, so aRc. Thus, we get

$$\bigwedge h_L(G) \le h_L(a) \le \bigvee U = \bigvee h_L(I),$$

where I is the round ideal generated by $\bigcup_{k=1}^n I_k$. By round-compactness, pick $d \in G \cap I$. By definition of G, pick $e \in F$ such that dRe, and by definition of I, pick a finite subset $B \subseteq \bigcup_{k=1}^n I_k$ such that $d \prec_L \bigvee B$. We then also get $\bigvee BRe$, so, since R is a j-morphism, there is a finite subset A of $BR(\_)$ such that $e \prec_M \bigvee A$. Now $\bigvee A \in F$ since $e \in F$, so that $x \le h_M(\bigvee A)$. Moreover, $h_M(\bigvee A) = \bigvee h_M(A) \le \bigvee R^\pi(U)$, because for each $a \in A$ there is some $b \in B \subseteq \bigcup_{k=1}^n I_k$ such that bRa. We conclude that $x \le \bigvee R^\pi(U)$, as required. $\qquad\square$

At this point, we have seen that R^π always preserves arbitrary meets and up-directed joins of round-ideal elements, and also finite joins of round-ideal elements in case R is a j-morphism. The following proposition follows immediately.

Proposition 2.4.16. *If R is a j-morphism, then R^π preserves arbitrary joins of round-ideal elements.*

It is natural to ask if we can prove by similar methods that R^π preserves *all* joins, as is possible for canonical extensions of lattices using a 'restricted distributivity' law (cf. [54], Lemma 3.2). Although we are able to prove an

analogue of that law for proximity lattices, we do not see at this point how to use it to generalize the proof from [54] that R^π preserves all joins. Instead, we will prove the result that R^π preserves all joins using the connection between duality and canonical extensions for proximity lattices, that we develop in the next section.

2.5. Duality for stably compact spaces

The dual equivalence between stably compact spaces and certain proximity lattices was established in [88], and reproved by categorical methods in Corollary 2.3.18 above. We first recall how the dual space is defined in [88], and then show in Proposition 2.5.6 that these two dual equivalences indeed associate the same dual space to any join-strong distributive proximity lattice. After this, we show in Theorem 2.5.7 that canonical extensions of distributive join-strong proximity lattices can be constructed using the dual space.

The points of the dual space of a distributive proximity lattice are the round filters of the lattice which are *prime*, in the following sense.

Definition 2.5.1. Let $(L, \prec)$ be a proximity lattice. A round filter $F \subseteq L$ is called *prime* if, for any finite set $A \subseteq L$, $\bigvee A \in F$ implies $A \cap F \neq \emptyset$.

The following theorem is the relevant consequence of the axiom of choice in our setting.

Theorem 2.5.2 (Prime round filter theorem). *Let $(D, \prec)$ be a join-strong distributive proximity lattice. Let G be a round filter and J a round ideal such that $G \cap J = \emptyset$. Then there exists a prime round filter F_0 such that $F_0 \cap J = \emptyset$ and $G \subseteq F_0$.*

Proof. Let $\mathcal{C} := \{F : F$ a round filter, $F \cap J = \emptyset, G \subseteq F\}$. Note that the union of a chain of round filters is a round filter. So, by Zorn's Lemma, we can take a maximal $F_0 \in \mathcal{C}$. It remains to show that F_0 is prime. Suppose, to obtain a contradiction, that there exist $d, e \in D$ such that $d \vee e \in F_0$, but $d \notin F_0$ and $e \notin F_0$. Consider the set $F_1 := \{x \mid d \vee x \in F_0\}$. Note that F_1 is a filter, by distributivity of D. Also note that F_1 is round, using join-strongness and idempotence of $\prec$. Moreover, F_1 contains e, so F_1 is a round filter strictly containing F_0. Therefore, by maximality of F_0, pick an element $a \in F_1 \cap J$. Now $F_2 := \{x \mid a \vee x \in F_0\}$ is again a round filter which strictly contains F_0, this time because $d \in F_2$. Therefore, pick an element $b \in F_2 \cap J$. Now $a \vee b \in F_0$ since $b \in F_2$, and $a \vee b \in J$ since J is an ideal. On the other hand, $F_0 \cap J = \emptyset$, so this is the desired contradiction. □

Definition 2.5.3. Let $(D, \prec)$ be a join-strong distributive proximity lattice. The *round spectrum* of $(D, \prec)$ is the space **spec** $(D, \prec)$, whose points are prime round filters of $(D, \prec)$ and whose topology is generated by the sets of the form $U_d := \{F : d \in F\}$, for $d \in D$.

The round spectrum can also be obtained via the duality between sober spaces and spatial frames (cf. Theorem 1.2.7).

Lemma 2.5.4 ([88], Corollary 12). *The round spectrum of a join-strong distributive proximity lattice is homeomorphic to the space of points of the arithmetic frame of round ideals.*

Example 2.5.5. If X is a stably compact space, and $(D, \prec)$ is an open-basis presentation of X, then a simple topological argument will show that prime round filters of D correspond to completely prime filters of open sets of X. It follows in particular that the round spectrum of any open-basis presentation of X is homeomorphic to X.

Proposition 2.5.6. *The dual space of a join-strong distributive proximity lattice $(D, \prec)$, according to the duality in Corollary 2.3.18, is homeomorphic to the round spectrum of $(D, \prec)$.*

Proof. The duality functor of Corollary 2.3.18 uses the fact that any join-strong distributive proximity lattice $(D, \prec)$ is a j-morphic retract of the lattice $(\prec\mathbf{idl}(D), \leq)$. The retraction j-morphisms R, S from the proof of Theorem 2.3.17 compose to the identity on the object $(D, \prec)$. One may now calculate from the definitions that the composition $R \circ S$ is the join-approximable relation $\gg : (\prec\mathbf{idl}(D), \leq) \nrightarrow (\prec\mathbf{idl}(D), \leq)$, i.e., the way-above relation of the frame $(\prec\mathbf{idl}(D), \leq)$ (also see Proposition 2.3.19). Now, by the duality of distributive lattices with join-approximable relation and spectral spaces with continuous functions (Theorem 2.2.6), the relation $\gg$ yields a continuous function f on the dual space of the lattice $(\prec\mathbf{idl}(D), \leq)$. The dual space associated to $(D, \prec)$ in Corollary 2.3.18 is the image of this map f. A short argument now shows that a prime filter G of the lattice $(\prec\mathbf{idl}(D), \leq)$ is in the image of f if, and only if, there exists a round prime filter F of $(D, \prec)$ such that $G = \{I \in \prec\mathbf{idl}(D) \mid F \cap I \neq \emptyset\}$. Using this fact, one easily shows that the round spectrum of $\prec\mathbf{idl}(D)$ is homeomorphic to the image of f. □

The canonical extension of a join-strong distributive proximity lattice can be constructed from the round spectrum. Specifically, we have the following result.

Theorem 2.5.7. *Let $(D, \prec)$ be a join-strong distributive proximity lattice. Let S be the complete lattice of saturated sets of the round spectrum of $(D, \prec)$. Then $h : D \to S$, defined by $d \mapsto \{F : d \in F\}$, is a π-canonical extension of $(D, \prec)$.*

Proof. It is not hard to see that h is a homomorphism. One may further show that the round-ideal elements of the extension $h : D \to S$ are exactly the open sets of the round spectrum, and that the round-filter elements are exactly the compact saturated sets of round spectrum. From this, round-denseness follows. For round-compactness, one uses the prime round filter theorem[4]. The fact that h is round-join-preserving is immediate from the definition of round filters.   □

The dual statement of Theorem 2.5.7 also holds, replacing join-strong by meet-strong and the prime round filter spectrum by the prime round ideal spectrum.

We now remark on how the special case of distributive lattices and spectral spaces fits in the general picture. We already observed in Proposition 2.4.9 that the π- and σ-extension of a doubly strong proximity lattice $(L, \prec)$ coincide if, and only if, the relation $\prec$ is reflexive. If the underlying lattice is distributive, this situation relates to the dual space being spectral, as follows.

Proposition 2.5.8. *Let $(D, \prec_D)$ be a distributive join-strong proximity lattice. The following are equivalent.*

1. *$(D, \prec_D)$ is j-isomorphic to some distributive proximity lattice of the form $(E, \leq_E)$,*

2. *$(D, \prec_D)$ is j-isomorphic to some distributive proximity lattice $(E, \prec_E)$ with $\prec_E$ reflexive,*

3. *The round spectrum of $(D, \prec_D)$ is a spectral space.*

Proof. It is trivial that (1) implies (2). For (2) implies (3), it suffices to show that if $\prec_D$ is reflexive, then the round spectrum of $(D, \prec_D)$ is spectral. A straight-forward application of the prime round filter theorem shows that each basic open set U_d is compact in this situation, so that $\{U_d\}_{d \in D}$ is a basis of compact open sets for the round spectrum. For (3) implies (1), let E be the distributive lattice of compact open sets of the round spectrum of $(D, \prec_D)$. By Stone duality for distributive lattices, the round spectrum of $(D, \prec_D)$ is homeomorphic to the round spectrum of $(E, \leq_E)$. Hence, the proximity lattices $(D, \prec_D)$ and $(E, \leq)$ must be j-isomorphic by duality for join-strong distributive proximity lattices.   □

Looking back at Proposition 2.4.9, we now see that in the distributive case, we can conclude something more from the assumption that the relation $\prec$ is reflexive.

---

[4]As a referee pointed out, one could alternatively prove round-compactness by invoking the fact that stably compact spaces are *well-filtered*, following the terminology of [67], p. 147.

Corollary 2.5.9. *If $(D, \prec)$ is a doubly strong distributive proximity lattice and $\prec$ is reflexive, then there is a distributive lattice E such that $(D, \prec)$ is both j- and m-isomorphic to $(E, \leq_E)$.*

Regarding the extensions of morphisms, recall that at the end of Section 2.4, the question whether the π-extension of a j-morphism preserves all joins remained open. We can now show that, for distributive lattices, the π-extension of a j-morphism R is concretely realized as the inverse image map of the continuous map to which R corresponds via the duality. It will follow in particular that R^π preserves all joins and meets.

More precisely, our set-up is as follows: let $R : (D, \prec) \rightarrow (E, \prec)$ be a j-morphism between join-strong distributive proximity lattices, and let f_R be the dual map from the round spectrum of $(E, \prec)$ to the round spectrum of $(D, \prec)$. For F a prime round filter of $(E, \prec)$, we have $f_R(F) = (\_)RF$. Let $h_D : (D, \prec) \rightarrow (D, \prec)^\pi$ and $h_E : (E, \prec) \rightarrow (E, \prec)^\pi$ be the π-canonical extensions, which, up to isomorphism, are the complete lattices of saturated sets of the spectra, as given in Theorem 2.5.7.

Proposition 2.5.10. *In the above setting, we have for all $u \in (D, \prec)^\pi$ that $R^\pi(u) = f_R^{-1}(u)$. In particular, R^π preserves all joins and meets.*

Proof. For an open set u of the space $\mathbf{spec}\,(D, \prec)$, we have $u = \bigvee h_D(I)$, where I is the round ideal $h_D^{-1}(\downarrow u)$. Hence,

$$
\begin{aligned}
f_R^{-1}(u) &= \{F \in \mathbf{spec}\,(E, \prec) : (\_)RF \cap I \neq \varnothing\} \\
&= \bigcup_{e \in IR(\_)} h_E(e) \\
&= \bigvee \{h_E(b) \mid \exists a \in D : h_D(a) \leq u, aRb\},
\end{aligned}
$$

so $f_R^{-1}(u)$ agrees with the definition of $R^\pi(u)$.
Now, for any saturated set s of the dual space of $(D, \prec)$, we have

$$
\begin{aligned}
f_R^{-1}(s) &= \{F \in \mathbf{spec}\,(E, \prec) \mid f(F) \in s\} \\
&= \{F \in \mathbf{spec}\,(E, \prec) \mid \forall u \text{ open, if } s \subseteq u \text{ then } f(F) \in s\} \\
&= \bigwedge \{f_T^{-1}(u) : s \leq u, u \in I((D, \prec)^\pi)\},
\end{aligned}
$$

where the last step follows from the proof of Theorem 2.5.7. So the definition of R^π completely agrees with the values of f_R^{-1}. $\qquad\square$

We make some final remarks about our last result. On the one hand, it shows that duality can be a powerful tool to answer questions which are algebraically difficult (cf. the proofs in Section 2.4). On the other hand, using the duality we can so far only prove results in the distributive setting,

whereas the results in Section 2.4 hold in arbitrary proximity lattices, for which no duality is available yet. Moreover, the duality makes essential use of the axiom of choice, which was avoided completely in Section 2.4.

Concluding remarks

In this chapter, we have shown that the theory of canonical extensions, which in the past has generated powerful results for logics based on lattices, can be extended to proximity lattices. The canonical extension gives an algebraic description of the complete lattice of saturated sets of a stably compact space, starting from any basis presentation of the space. We now relate our work to the literature, and point to directions for future work.

Our definition of proximity lattice is close to, but a bit more general than that of Jung and Sünderhauf [88]. Note in particular that in [88], all proximity lattices were distributive, but this assumption is not necessary for a large part of the theory of canonical extensions, developed here in Section 2.4. Also, in the original definition of Jung and Sünderhauf, it was emphasized that proximity lattices are not necessarily increasing. However, as we showed in Proposition 2.3.19, the category of join-strong proximity lattices is equivalent to its full subcategory that only consists of the increasing ones. The assumption that the proximity relation *is* increasing makes the ensuing theory quite a bit cleaner and easier to present. Moreover, compared to [88], our morphisms are going in the opposite direction. We have made this choice because we wanted the category of algebras to be *dually* equivalent to the category of spaces; this way, the dual equivalence between the categories of join-strong distributive proximity lattices and stably compact spaces directly generalizes Stone duality between distributive lattices and spectral spaces. Of course, the choice of direction of morphisms is ultimately a matter of taste, especially because the morphisms in the category are relations.

The relation between join-strong and meet-strong proximity lattices was clarified in Example 2.3.8: a meet-strong representation of a stably compact space X corresponds to a join-strong representation of the co-compact dual X^{∂}. We thus observe that the *doubly* strong proximity lattices from Jung and Sünderhauf [88] simultaneously represent *both* the space X and its co-compact dual X^{∂}. This is the reason that a rather complicated construction, involving pairs of open and compact sets, was needed in [88] in order to obtain the representing lattice from a space (cf. Example 2.3.5 above). By contrast, we used open-basis presentations to represent a space X by a proximity lattice, which are not doubly strong, but only join-strong. We thus separate the issue of representing X from representing its co-compact dual

X^∂. In this respect, our work is closer in spirit to the work of Smyth [135], where proximity lattices where originally introduced.

Gehrke and Vosmaer [66] express the canonical extension of a lattice as a *dcpo presentation* (also see [87]). The same can be done for our π- and σ-canonical extensions of proximity lattices, via a straightforward generalisation of the methods used in [66]. In that work, the big advantage in presenting the canonical extension via dcpo presentation was that it shed new light on the preservation of inequalities: known results about dcpo presentations and dcpo algebras were applied to obtain powerful results about the preservation of inequalities in the canonical extension. We expect that similar methods would apply in our setting, if one were to study the canonicity of inequations in proximity lattices. We leave this as a topic for further research.

Another obvious direction for future work is to apply the canonical extension of proximity lattices to logic and, more specifically, to the theory of probabilistic programming. One of the original motivations for the work of Jung and Sünderhauf [88] was to develop a 'continuous' version of Domain Theory in Logical Form (cf. [86]). Just as the 'classical' canonical extension proved to be a powerful tool in modal logic, we expect that canonical extensions of proximity lattices could prove to be useful in the study of the Multi-lingual Sequent Calculus developed in a sequence of papers by Kegelman, Moshier, Jung and Sünderhauf (e.g., [90], [119], [86]).

The definition of the canonical extension for proximity lattices opens up a variety of new questions regarding the behaviour of canonical extensions of maps between proximity lattices, in the line of [56], [57] and [54]. We believe that we have only scratched the surface of what can be said about these questions, by showing some properties of the canonical extension of proximity morphisms in Section 2.4.

We saw in Section 2.4 that join-strong proximity lattices have π-extensions, meet-strong proximity lattices have σ-extensions, and thus a doubly strong proximity lattice has both extensions. Moreover, these two extensions coincide if, and only if, the additional relation is reflexive. This sheds new light on a part of the classical theory of canonical extensions that was perhaps not so well understood in the past, namely that even though a lattice has *one* canonical extension, there are *two* ways to extend a lattice map, i.e., a σ- and a π-version. In light of our work in this chapter, we would explain this phenomenon by saying that a lattice in principle does have both a σ- and a π-extension, but that these two extensions coincide because of the reflexivity of the lattice order.

An important related issue in the theory of proximity lattices and their canonical extensions that we think needs to be explored further is that of order-duality. More precisely, the power of canonical extensions lies in their

ability to deal with additional operations on the lattice which may be order-preserving or order-reversing. It is thus important to understand the role of order-duality in more detail, although that was not the focus of this chapter. For the same reason, it also seems natural to wonder whether there is a natural, more symmetrical notion of extension for those proximity lattices which do not satisfy any of the 'strong' axioms.

In Section 2.5, we have put the Jung and Sünderhauf duality from [88] in a categorical perspective. A question which remains open is whether it is possible, as it was in the classical case [139], to remove the requirement of distributivity on the underlying lattice. This would involve replacing the spectrum of prime round filters by the space of maximal pairs of round filters and round ideals, in a sense which we believe could be made precise in future work.

A last issue that we would like to deal with in future work is how essential the use of duality is to prove Propositions 2.5.8 and 2.5.10. Concretely, we would expect that versions of Propositions 2.5.8 and 2.5.10 hold in which we no longer need to refer to the dual space, and we therefore also avoid the axiom of choice and the assumption that the underlying lattices are distributive.

Chapter 3. Duality for sheaves of distributive-lattice-ordered algebras

In this chapter, we study sheaf representations of distributive lattices using Priestley duality. We introduce a notion for sheaves over stably compact spaces of being flasque on a given basis. We prove that sheaves of distributive lattices over a stably compact space that are flasque on a basis dually correspond to decompositions of the Priestley dual space of the lattice of global sections, which are fibred over the co-compact dual of the base space. Moreover, this result extends in a straightforward way to distributive-lattice-ordered algebras, for which it then also provides a tool for studying sheaf representations.

Since a distributive lattice A is completely determined by its Priestley dual space X, it is reasonable to expect that a sheaf representation of A also corresponds to a certain kind of representation of the space X. Indeed, sheaf representations of distributive lattices over a Boolean space can be understood dually as disjoint decompositions of the Priestley and Stone dual spaces of the distributive lattice in question [76, 49], also cf. Remark 3.2.3 below.

In practice (cf., e.g., Chapter 4), one is often interested in sheaves over base spaces which are not Boolean, but for example only compact Hausdorff, or spectral. We therefore place ourselves in the wider context of sheaves over stably compact spaces (cf. Chapter 2). We will show that certain sheaf representations of a distributive lattice A over a stably compact base space Y dually correspond to certain continuous maps from the dual Priestley space X of A to the co-compact dual Y^∂ of Y. To be a bit more precise, our results will be parametric in the choice of a basis B for the open sets of the stably compact space Y. A sheaf will be called B-flasque, or flasque on the basis B, if each section over a basic open set extends to a global section. We will identify the relevant property for a continuous map from X to Y^∂ to be dual to a B-flasque sheaf; such maps will be called B-patching decompositions. Our main theorem is then the following.

Theorem 3.3.7. *Let A be a distributive lattice with dual Priestley space X, and let Y be a stably compact space with lattice basis B. The B-flasque sheaves over Y with lattice of global sections A are in one-to-one correspondence with the B-patching decompositions of the space X over Y^∂.*

Outline of the chapter. We first recall some preliminaries on sheaves of distributive lattices and stably compact spaces in Section 3.1. In Section 3.2

we show that any *B*-flasque sheaf *F* gives rise to a lattice homomorphism from *B*^{op} to the congruence lattice of the lattice of global sections of *F*. We subsequently show that such a lattice homomorphism can be lifted to obtain a frame homomorphism from the open sets of the co-compact dual of the base space to the frame of congruences. From there, we obtain a continuous decomposition. In Theorem 3.2.7, we concretely describe this decomposition in terms of the sheaf of departure. In Section 3.3, we then identify the decompositions which correspond to *B*-flasque sheaves (Proposition 3.3.6), allowing us to prove our main theorem, Theorem 3.3.7. We end the chapter by giving an application to the dual spaces of products, and we discuss how Theorem 3.3.7 can also be applied to sheaves of distributive lattices *with additional operations*, as this will be important for applications in Chapter 4.

3.1. Preliminaries on sheaves and topology

We briefly review the well-known correspondence between sheaves and étalé spaces. We refer to [108, Chapter 2] for further background. Here, we give some details for the particular case of sheaves of distributive lattices.

Definition 3.1.1. Let Y be a topological space. A *presheaf of distributive lattices* over Y is a functor $F : \Omega(Y)^{\mathrm{op}} \to \mathbf{DL}$. If $U \subseteq V$, then the map $F(U \subseteq V) : F(V) \to F(U)$ is denoted by $(-)|_{V,U}$, or also by $(-)|_U$. A presheaf F of distributive lattices is a *sheaf* if, for any collection of open sets $(U_i)_{i \in I}$ and for any collection $(s_i)_{i \in I}$ with $s_i \in F(U_i)$, if $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $i, j \in I$, then there exists a unique $s \in F(\bigcup_{i \in I} U_i)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

Let F be a presheaf of distributive lattices over Y. For $y \in Y$, the *stalk of F at y*, F_y, is the colimit of the directed system of homomorphisms $((-)|_U : F(V) \to F(U))_{y \in U \subseteq V}$. Let E be the disjoint union of the stalks of F and let $p : E \to Y$ be the map sending $e \in F_y$ to y. Note that, for any $U \in \Omega(Y)$, each element s of $F(U)$ determines a function $\dot{s} : U \to E$, by letting $\dot{s}(y)$ be the image of $s \in F(U)$ under the colimit map $F(U) \to F_y$. Let σ be the topology on E generated by taking all the images of these functions $\dot{s}$, where U ranges over $\Omega(Y)$ and s ranges over $F(U)$, as a subbasis for the open sets. One may then show that $p : E \to Y$ is a *local homeomorphism*, i.e., that p is continuous and every point of E has an open neighbourhood U such that $p(U)$ is open and $p : U \to p(U)$ is a homeomorphism. The set E, equipped with this topology σ, is called the *étalé space of germs* associated to F. Since each stalk F_y is a distributive lattice, the étalé space E is naturally equipped with partial binary operations $\vee, \wedge$ whose domain is

$E \times_Y E := \{(e, e') \in E \mid p(e) = p(e')\}$, and two functions $0 : Y \to E$ and $1 : Y \to E$, which send a base point $y \in Y$ to, respectively, the elements 0_y and 1_y of F_y. It is straightforward to verify that $\vee, \wedge, 0, 1$ are continuous (here, for $\vee$ and $\wedge$, the topology on $E \times_Y E$ is the topology inherited from the product $E \times E$). We refer the reader to, e.g., [70, II.1.2] or [108, II.7] for more details on this construction.

Definition 3.1.2. An *étalé space of distributive lattices* over a base space Y is a space E with a continuous map $p : E \to Y$ such that (i) p is a local homeomorphism; (ii) for each $y \in Y$, the set $p^{-1}(y)$ is equipped with a distributive lattice structure; (iii) each of the maps $\vee : E \times_Y E \to E$, $\wedge : E \times_Y E \to E$, $0 : Y \to E$, $1 : Y \to E$ is continuous.

Given an étalé space $p : E \to Y$ of distributive lattices, one may associate to it the *sheaf F of local sections*: for each open set U of Y, $F(U)$ is defined to be the sublattice of $\prod_{y \in U} p^{-1}(y)$ consisting of those elements s which are continuous when viewed as maps $s : U \to E$. If U, V are open sets in Y and $U \subseteq V$, then the map $(-)|_U : F(V) \to F(U)$ is defined as the restriction of the projection $\prod_{y \in V} p^{-1}(y) \to \prod_{y \in U} p^{-1}(y)$. In this context, the set $F(Y)$ is called the lattice of *global sections* of the étalé space E. One may now show that F is indeed a sheaf of distributive lattices in the sense of Definition 3.1.1. For easy reference throughout this chapter, we record the following facts as a theorem (cf., e.g., [108, II.6] for the proof).

Theorem 3.1.3. *The constructions of the sheaf of local sections and the étalé space of germs are mutually inverse up to isomorphism. In particular, for any sheaf F of distributive lattices over Y, the distributive lattice $F(Y)$ is isomorphic to the lattice of global sections of the étalé space of germs of F.*

Notation. Throughout this chapter, unless otherwise mentioned, whenever we use the word "sheaf" or "étalé space", this should be understood to mean "sheaf of distributive lattices" or "étalé space of distributive lattices".

We now prove two preliminary facts about about stably compact spaces that we will use in this chapter. The first is well-known.

Lemma 3.1.4. *Let Y be a stably compact space with specialization order $\leq$. If $K \subseteq Y$ is compact in the co-compact dual topology Y^∂, then $\uparrow K$ is closed in Y.*

Proof. Any Y^∂-open cover of K is also a Y^∂-open cover of $\uparrow K$, since Y^∂-open sets are upsets. Therefore, $\uparrow K$ is compact in Y^∂, and it is also saturated in Y^∂, since it is an upset in the specialization order of Y. We conclude that $\uparrow K$ is compact-saturated in Y^∂, so it is closed in Y by Theorem 2.1.8 in Chapter 2. $\qquad\square$

The last simple lemma, which shows how a lattice basis of a stably compact space interacts with the co-compact dual topology, will be crucial in the proof of Proposition 3.2.4.

Lemma 3.1.5. *Let Y be a stably compact space and let B be a lattice basis for the open sets of Y. For any open set W in Y^∂, if $y \in W$, then there exists a Y^∂-open set V and a basic Y-open set $U \in B$ such that $y \in V \subseteq U^c \subseteq W$.*

Proof. Let W be an open set in Y^∂ and $y \in W$. Since Y^∂ is a locally compact space, there exist a Y^∂-open set V and a Y^∂-compact-saturated set K such that $y \in V \subseteq K \subseteq W$. Since K^c is open in Y, $W^c \subseteq K^c = \bigcup \{U \in B \mid U \subseteq K^c\}$. Since W is open in Y^∂, the set W^c is compact in Y, so, since B is a lattice basis, there exists $U \in B$ such that $W^c \subseteq U \subseteq K^c$. Taking complements, we see that $K \subseteq U^c \subseteq W$. In particular, we have $V \subseteq U^c \subseteq W$, as required.   □

This last lemma is essentially known and has been used in the literature on stably compact spaces (e.g., [88], [100]), but we have not been able to find a source where it is stated and proved explicitly in this form.

3.2. Decompositions from sheaves

In this chapter, we will work with sheaves that are flasque on a basis. This concept generalizes the notion of flasque (or flabby) sheaf from the literature (cf., e.g., [70, Section II.3.1] or [18, Section II.5]).

Definition 3.2.1. Let F be a sheaf on a topological space Y, and let B be a basis of open sets for the space Y. We say that F is *flasque on B*, or *B-flasque* if, for every $U \in B$, the restriction map $F(Y) \to F(U)$ is surjective.

With this definition, a sheaf F is flasque in the traditional sense of the word if, and only if, F is "$\Omega(Y)$-flasque", where $\Omega(Y)$ is the collection of all open sets of Y.

If $p : E \to Y$ is the étale map corresponding to a sheaf F, then F is B-flasque if, and only if, any continuous section s over a basic open set $U \in B$ can be extended to a global continuous section.

Suppose F is a sheaf over a base space Y which is flasque on a basis B for the space Y. The assignment

$$U \mapsto \vartheta_F(U) := \ker(F(Y) \xrightarrow{(-)|_U} F(U)), \quad (U \in B) \tag{3.1}$$

naturally defines a function ϑ_F from B to the frame $\mathrm{Con}(F(Y))$ of congruences of the distributive lattice $F(Y)$.

Proposition 3.2.2. *Let F be a sheaf over a base space Y which is flasque on a lattice basis B for the space Y. Then the function ϑ_F defined in (3.1) is a homomorphism from B^{op} to $\mathrm{Con}(F(Y))$, and any two congruences in the image of ϑ_F permute.*

Proof. Note that $\vartheta_F : B^{\mathrm{op}} \to \mathrm{Con}(F(Y))$ is order-preserving. This follows from the fact that, for any $U, U' \in B$ with $U \subseteq U'$, the following diagram commutes:

$$
\begin{array}{ccc}
 & F(Y) & \\
\scriptstyle |_{U'} \swarrow & & \searrow \scriptstyle |_{U} \\
F(U') & \xrightarrow{\quad |_{U',U} \quad} & F(U)
\end{array}
$$

We now prove that ϑ_F sends unions in B to intersections in $\mathrm{Con}(F(Y))$. Let $\mathcal{U} \subseteq B$ be a finite subset. Since ϑ_F is order-preserving, we clearly have that $\vartheta_F(\bigcup \mathcal{U}) \subseteq \bigcap_{U \in \mathcal{U}} \vartheta_F(U)$. For the other inclusion, suppose that $(a,b) \in \vartheta_F(U)$ for all $U \in \mathcal{U}$. Then both $a|_{\bigcup \mathcal{U}}$ and $b|_{\bigcup \mathcal{U}}$ are patches of the compatible family $(a|_U)_{U \in \mathcal{U}} = (b|_U)_{U \in \mathcal{U}}$, and therefore they are equal by the uniqueness part of the sheaf property. Thus, $(a,b) \in \vartheta_F(\bigcup \mathcal{U})$, as required. We now prove that ϑ_F sends finite intersections in B to finite joins in $\mathrm{Con}(F(Y))$. Clearly, $\vartheta_F(Y) = \Delta = \bot_{\mathrm{Con}(F(Y))}$, so ϑ_F sends the empty intersection to the empty join. Now let $U_1, U_2 \in B$. It suffices to prove that the following inclusions hold:

$$\vartheta_F(U_1) \vee \vartheta_F(U_2) \subseteq \vartheta_F(U_1 \cap U_2) \subseteq \vartheta_F(U_1) \circ \vartheta_F(U_2). \qquad (3.2)$$

Then, since $\vartheta_F(U_1) \circ \vartheta_F(U_2) \subseteq \vartheta_F(U_1) \vee \vartheta_F(U_2)$ always holds in the congruence lattice, we will have equality throughout, as required. Note that this argument shows in particular that any two congruences in the image of ϑ_F permute. We now prove the inclusions in (3.2). The first inclusion is clear from the fact that ϑ_F is order-preserving. For the second inclusion, suppose that $(a,b) \in \vartheta_F(U_1 \cap U_2)$. Then $\{a|_{U_1}, b|_{U_2}\}$ is a compatible family. Thus, by the existence part of the sheaf property, pick $c \in F(U_1 \cup U_2)$ such that $c|_{U_1} = a|_{U_1}$ and $c|_{U_2} = b|_{U_2}$. Since $U_1 \cup U_2 \in B$ and F is B-flasque, the map $F(Y) \to F(U_1 \cup U_2)$ is surjective. Pick $d \in F(Y)$ such that $d|_{U_1 \cup U_2} = c$. We now have that $d|_{U_1} = c|_{U_1} = a|_{U_1}$, so $(a,d) \in \vartheta_F(U_1)$, and similarly $(d,b) \in \vartheta_F(U_2)$. Therefore, $(a,b) \in \vartheta_F(U_1) \circ \vartheta_F(U_2)$, as required. $\qquad \square$

Remark 3.2.3. We note that Proposition 3.2.2 in particular implies well-known results on sheaves over Boolean spaces from [20], [30], as follows. Suppose that Y is a Boolean space. Notice that any sheaf F with at least one global section is flasque on the canonical basis B of clopen sets for Y.

It now follows from Proposition 3.2.2 that, for any clopen set $K \in B$, the congruence $\vartheta_F(K)$ has a complement $\vartheta_F(K^c)$ in the lattice $\mathrm{Con}(F(Y))$, with which it permutes. Thus, ϑ_F is a homomorphism from B^{op} to the Boolean algebra of factor congruences of $F(Y)$. Now, the Boolean algebra of factor congruences of $F(Y)$ is a subalgebra of the Boolean algebra of clopen subsets of the Priestley dual space X of the lattice $F(Y)$ (cf. Example 1.1.13 in Chapter 1). By Stone duality for Boolean algebras, the homomorphism ϑ_F now corresponds to a continuous function $q_F : X \to Y$. The function q_F gives a disjoint decomposition of X, indexed over Y, where each set in the decomposition is an order component of X (cf. [76, 49]). Note that, for each $y \in Y$, the closed subspace $q_F^{-1}(y)$ of X is dual to the stalk A_y of F at y.

In the rest of this chapter, we generalize the construction in Remark 3.2.3 to the context where Y is an arbitrary stably compact space, instead of a Boolean space. To this end, we will first show that, if Y is a stably compact space, then the homomorphism ϑ_F can be lifted to a frame homomorphism from the open sets of the co-compact dual Y^∂ to the frame of congruences of $F(Y)$. This obviously follows from the following, slightly more general, result.

Proposition 3.2.4. *Suppose that B is a lattice basis for the open sets of a stably compact space Y and that $h : B^{\mathrm{op}} \to M$ is a lattice homomorphism from B^{op} into a frame M. Then the function $\widetilde{h} : \Omega(Y^\partial) \to M$ defined by*

$$\widetilde{h}(W) := \bigvee \{h(U) \mid U \in B, \ U^c \subseteq W\}$$

is a frame homomorphism.

Proof. It is clear that $\widetilde{h}$ is order-preserving. Note that $\widetilde{h}$ preserves the empty intersection, since $\widetilde{h}(Y) = \bigvee \{h(U) \mid U \in B, \ U^c \subseteq Y\} \geq h(\emptyset) = \top$. Using that M is a frame and that h is a homomorphism from B^{op} to M, a straightforward calculation shows that $\widetilde{h}$ preserves binary meets. To prove that $\widetilde{h}$ preserves arbitrary joins, let $(W_i)_{i \in I}$ be a collection of open sets of Y^∂. We need to show that $\widetilde{h}(\bigcup_{i \in I} W_i) \leq \bigvee_{i \in I} \widetilde{h}(W_i)$, as the other inequality is clear from the fact that $\widetilde{h}$ is order-preserving. Let $U_0 \in B$ be arbitrary such that $(U_0)^c \subseteq \bigcup_{i \in I} W_i$. We need to show that $h(U_0) \leq \bigvee_{i \in I} \widetilde{h}(W_i)$. By Lemma 3.1.5, for each $i \in I$ we have that

$$W_i = \bigcup \{V \in \Omega(Y^\partial) \mid \text{ there exists } U \in B \text{ such that } V \subseteq U^c \subseteq W_i\}.$$

Hence, the collection $\mathcal{C} := \{V \in \Omega(Y^\partial) \mid \exists i \in I, U \in B : V \subseteq U^c \subseteq W_i\}$ is an open cover of the set $(U_0)^c$. Since U_0 is open in Y, its complement $(U_0)^c$ is compact in Y^∂. Let $\mathcal{F} \subseteq \mathcal{C}$ be a finite subcover. For each $V \in \mathcal{F}$,

pick $i_V \in I$ and $U_V \in B$ such that $V \subseteq (U_V)^c \subseteq W_{i_V}$. We then have $(U_0)^c \subseteq \bigcup_{V \in \mathcal{F}} V \subseteq \bigcup_{V \in \mathcal{F}} (U_V)^c$, so that $\bigcap_{V \in \mathcal{F}} U_V \subseteq U_0$. Also, for each $V \in \mathcal{F}$, we have that $(U_V)^c \subseteq W_{i_V}$, so that $h(U_V) \leq \widetilde{h}(W_{i_V})$, by definition of $\widetilde{h}$. Therefore, since h is a homomorphism from B^{op} to F, we have that $h(U_0) \leq \bigvee_{V \in \mathcal{F}} h(U_V) \leq \bigvee_{i \in I} \widetilde{h}(W_i)$, as required. $\qquad \square$

Combining Propositions 3.2.2 and 3.2.4, we conclude:

Theorem 3.2.5. *Let F be a sheaf over a stably compact space Y which is flasque on a lattice basis B for Y. Then the function $\widetilde{\vartheta_F}$ defined by*

$$\widetilde{\vartheta_F}(W) := \bigvee \{\ker(F(Y) \to F(U)) \mid U \in B, U^c \subseteq W\}$$

is a frame homomorphism from $\Omega(Y^\partial)$ to $\mathrm{Con}(F(Y))$.

We warn the reader that the function $\widetilde{\vartheta_F}$ is not an extension of the function ϑ_F defined in (3.1), at least not in the naive sense of the word 'extension'. However, note that if U is a compact-open set in Y, then U will always be in the lattice basis B, and the complement U^c of U will be open in Y^∂. It then follows easily from the definitions that, for U compact-open in Y, we have $\widetilde{\vartheta_F}(U^c) = \vartheta_F(U)$.

Remark 3.2.6. The function $\widetilde{\vartheta_F}$ in this theorem generalizes the construction of the stalk of a sheaf, in the following sense. Let $\leq$ denote the specialization order of Y. For any point $y \in Y$, the set $W_y := (\downarrow y)^c$ is an open set in Y^∂. By definition, $\widetilde{\vartheta_F}(W_y)$ is the smallest congruence containing each congruence $\vartheta_F(U) = \ker(F(Y) \to F(U))$, where $U \in B$ and $y \in U$. By the definition of stalk and the fact that F is B-flasque, this congruence is precisely the kernel of the restriction map $F(Y) \twoheadrightarrow F_y$ onto the stalk of F at y. Hence, the quotient $F(Y) \twoheadrightarrow F(Y)/\widetilde{\vartheta_F}(W_y)$ is isomorphic to the stalk quotient $F(Y) \twoheadrightarrow F(Y)_y$.

Write $A := F(Y)$ for the distributive lattice of global sections of F, and X for the Priestley dual space of A. Recall from Priestley duality (Proposition 1.1.12) that there is a frame isomorphism $\varphi : (\mathrm{Con}(A), \subseteq) \to (\mathrm{Cl}(X), \supseteq)$, which sends a congruence ϑ to the closed subset $\varphi(\vartheta)$ of X, consisting of those $x \in X$ such that, for all $(a, b) \in \vartheta$, $x \in \widehat{a} \leftrightarrow x \in \widehat{b}$. For $y \in Y$, we write X_y for the closed subspace corresponding, via φ, to the kernel of the stalk quotient $A \twoheadrightarrow A_y$. Thus, from the sheaf F, we obtain a relation $R_F \subseteq X \times Y$, defined by $x R_F y \iff x \in X_y$. In the following theorem, we show that this relation is the upset of the graph of a continuous function from X to Y^∂.

Theorem 3.2.7. *There exists a unique continuous function $q_F : X \to Y^\partial$ such that the following diagram commutes:*

$$
\begin{array}{ccc}
 & \Omega(Y^\partial) & \\
\widetilde{\vartheta_F} \nearrow & & \searrow (q_F)^{-1} \\
\mathrm{Con}(A) & \xrightarrow{\;(-)^c \circ \varphi\;} & \Omega(X)
\end{array}
$$

Moreover, for any $x \in X$, $y \in Y$, we have $q_F(x) \leq y$ if, and only if, $x \in X_y$.

Proof. Write h_F for the composite $(-)^c \circ \varphi \circ \widetilde{\vartheta_F} : \Omega(Y^\partial) \to \Omega(X)$, which is a frame homomorphism by Theorem 3.2.5 and the fact that $(-)^c \circ \varphi$ is a frame isomorphism. By the duality between sober spaces and spatial frames (cf. Theorem 1.2.7), there exists a unique continuous function $q_F : X \to Y^\partial$ such that $(q_F)^{-1} = h_F$, i.e., such that the diagram commutes. Write $W_y := (\downarrow y)^c$ for $y \in Y$. Now note that, for any $x \in X$ and $y \in Y$,

$$
q_F(x) \leq y \iff x \notin (q_F)^{-1}(W_y) = h_F(W_y) \iff x \in \varphi(\widetilde{\vartheta_F}(W_y)) = X_y.
$$

Here, the equality $\varphi(\widetilde{\vartheta_F}(W_y)) = X_y$ follows from Remark 3.2.6. $\square$

Note that, by this theorem, for each $x \in X$, the value of $q_F(x)$ is the minimum of the set $x R_F(\_) = \{y \in Y \mid x \in X_y\}$ with respect to the specialization order of Y. If Y is a Boolean space, then q_F defined here coincides with the function q_F from Remark 3.2.3. More generally, if Y is a compact Hausdorff space, then $Y = Y^\partial$ and the specialization order is trivial, so that the set $x R_F(\_)$ is a singleton for each $x \in X$. Thus, in this case, q_F is a continuous function from X to Y which decomposes X as a disjoint union of closed subspaces X_y. In the general case where Y is a stably compact space, the subspaces X_y need not be mutually disjoint (indeed, as we will see shortly, the subspaces can even be contained one in the other), but we will still think of q_F as a (generalized, ordered) "decomposition". All this inspires the following definition.

Definition 3.2.8. We call the continuous map $q_F : X \to Y^\partial$ defined in Theorem 3.2.7 the *decomposition of the Priestley space X associated to the sheaf F.*

We now use the decomposition q_F to describe the closed subspaces of X that dually correspond to the quotients $A \twoheadrightarrow A_y$, for $y \in Y$, and $A \twoheadrightarrow F(U)$, for $U \in B$.

Proposition 3.2.9. *Let* $q_F : X \to Y^\partial$ *be the decomposition associated to the sheaf F.*

1. *For any* $y \in Y$, *the dual of the quotient* $A \twoheadrightarrow A_y$ *is the closed subspace* $(q_F)^{-1}(\downarrow y)$.

2. *For any* $U \in B$, *the dual of the quotient* $A \twoheadrightarrow F(U)$ *is the closed subspace* $\overline{(q_F)^{-1}(U)}$.

Proof. Note that it follows from the theorem that, for any $x \in X$ and $y \in Y$, we have $x \in X_y$ if, and only if, $q_F(x) \leq y$ in the specialization order of Y. Hence, $(q_F)^{-1}(\downarrow y) = X_y$, and the latter is by definition the closed subspace dual to the quotient $A \twoheadrightarrow A_y$. For the second item, note that the kernel of $A \twoheadrightarrow F(U)$ is the intersection of the kernels of $A \twoheadrightarrow A_y$, where y ranges over U. By Priestley duality (Proposition 1.1.12), the dual of the quotient $A \twoheadrightarrow F(U)$ is therefore equal to $\overline{\bigcup_{y \in U} X_y}$. By the first item, $X_y = (q_F)^{-1}(\downarrow y)$. Since open sets are downsets, we have $U = \bigcup_{y \in U} \downarrow y$, from which the result follows. $\square$

In particular, if U is compact-open in Y, then $U \in B$ and U is closed in Y^∂. By continuity of q_F, it then follows that $(q_F)^{-1}(U)$ is closed. So, in this case, by Proposition 3.2.9.2, the dual of the quotient $A \twoheadrightarrow F(U)$ is equal to $(q_F)^{-1}(U)$.

Proposition 3.2.9 shows that the essential information of the sheaf F can be reconstructed from the decomposition q_F. In the next section, we will give necessary and sufficient conditions for a continuous map $X \to Y^\partial$ to be the decomposition of X corresponding to some sheaf F.

3.3. Sheaves associated to decompositions

Notation. Throughout this section, X always denotes a Priestley space with dual distributive lattice A, and Y denotes a stably compact space, whose specialization order we denote by $\leq$.

We now describe the reverse of the process described in the previous section: from an arbitrary decomposition, we can define a sheaf. Suppose that $q: X \to Y^\partial$ is a continuous map to the co-compact dual of Y. With the aim of associating an étalé space to q, we observe the following. The set $\downarrow y$ is closed in Y^∂ for any $y \in Y$, whence $X_y := q^{-1}(\downarrow y)$ is closed. Let $f_y: A \twoheadrightarrow A_y$ be the quotient corresponding to X_y via Proposition 1.1.12, so that the kernel of f_y is the congruence ϑ_y defined by

$$a \, \vartheta_y \, b \iff \widehat{a} \cap X_y = \widehat{b} \cap X_y.$$

Now, for any $y, z \in Y$, if $y \leq z$ then $\vartheta_z \subseteq \vartheta_y$. (Indeed, $y \leq z$ entails $\downarrow y \subseteq \downarrow z$, so $X_y \subseteq X_z$. By Proposition 1.1.12, this is equivalent to $\vartheta_z \subseteq \vartheta_y$.) Thus, whenever $y \leq z$, we have a quotient map $f_{z,y} \colon A_z \twoheadrightarrow A_y$ which corresponds under Priestley duality to the subspace inclusion $X_y \hookrightarrow X_z$, namely, the unique map such that $f_{z,y} \circ f_z = f_y$.

We now standardly construct an étalé space over Y whose stalks are the distributive lattices A_y. Let E_q denote the disjoint union of the sets A_y, for $y \in Y$, and let $p \colon E_q \to Y$ be the map $[a]_{\vartheta_y} \in A_y \mapsto y \in Y$. The unique étalé topology on E_q can be described as follows. Any $a \in A$ has an associated global section $s_a \colon Y \to E_q$ of p that acts by $y \in Y \mapsto [a]_{\vartheta_y} \in A_y$. Let σ be the topology on E_q generated by the subbasis $\{s_a(U) \mid a \in A, \, U \text{ open in } Y\}$.

Proposition 3.3.1. *If E_q is equipped with the topology σ, then for any $a \in A$, the section $s_a \colon Y \to E_q$ is continuous, and $p \colon E_q \to Y$ is an étalé space of distributive lattices.*

Proof. For the first statement, it suffices to show that the inverse image under s_a of a subbasic set is open in Y. To this end, let $b \in A$ and U open in Y, and consider the inverse image $s_a^{-1}(s_b(U))$. Writing $\triangle$ to denote set-theoretic symmetric difference, for any $y \in Y$ we have:

$$y \in s_a^{-1}(s_b(U)) \iff y \in U \text{ and } [a]_{\vartheta_y} = [b]_{\vartheta_y},$$
$$\iff y \in U \text{ and } \widehat{a} \cap X_y = \widehat{b} \cap X_y,$$
$$\iff y \in U \text{ and } \forall x \in X_y : x \notin \widehat{a} \triangle \widehat{b},$$
$$\iff y \in U \cap (\uparrow q(\widehat{a} \triangle \widehat{b}))^c.$$

(For the last equivalence, recall that $x \in X_y$ iff $q(x) \leq y$ by definition.) Thus, the set $s_a^{-1}(s_b(U))$ is equal to $U \cap (\uparrow q(\widehat{a} \triangle \widehat{b}))^c$. We now show that $(\uparrow q(\widehat{a} \triangle \widehat{b}))^c$ is open. Since $\widehat{a} \triangle \widehat{b}$ is closed in the Priestley space X, it is compact. By continuity of q, $q(\widehat{a} \triangle \widehat{b})$ is compact in Y^∂. By Lemma 3.1.4, $\uparrow q(\widehat{a} \triangle \widehat{b})$ is closed in Y, so its complement is open. This concludes the proof that s_a is continuous. Now note that p is continuous: if $U \subseteq Y$ is open, then $p^{-1}(U) = \bigcup_{a \in A} s_a(U)$, which is open in the topology σ. To see that p is a local homeomorphism, let $e \in E_q$. Pick $a \in A$ and $y \in Y$ such that $e = [a]_{\vartheta_y}$. Then $V := s_a(Y)$ is an open set around e, and s_a is a continuous inverse to $p|_V$. Each set $p^{-1}(y) = A_y$ is equipped with the structure of a distributive lattice. It remains to show that all the operations are continuous in the sense of Definition 3.1.2. Since $0, 1 \in A$, the functions $s_0 : Y \to E_q$ and $s_1 : Y \to E_q$ are continuous. Now, to show that $\vee : E_q \times_Y E_q \to E_q$ is continuous, let $a \in A$ and U open in Y, and suppose that $(e, e') \in E_q \times_Y E_q$ such that $e \vee e' \in s_a(U)$. By definition of $E_q \times_Y E_q$ and s_a, pick $y \in Y$ and $b, b' \in A$ such that $e = f_y(b)$, $e' = f_y(b')$ and $f_y(b) \vee f_y(b') = f_y(a)$. Now

we have $b \vee b' \in A$, so the function $s_{b \vee b'}$ is continuous, and therefore the set $V := (s_{b \vee b'})^{-1}(s_a(U))$ is open in Y. One now easily checks that the function $\vee$ maps the open neighbourhood $(s_b(V) \times s_{b'}(V)) \cap (E_q \times_Y E_q)$ of (e, e') entirely into $s_a(U)$, so that $\vee$ is continuous. The proof that $\wedge$ is continuous is the same. $\qquad \square$

Definition 3.3.2. For $q \colon X \to Y^\partial$ a continuous map from a Priestley space X to the co-compact dual of a stably compact space Y, we call the space (E_q, σ) with the map $p \colon E_q \to Y$ defined above the *étalé space associated to* q. The corresponding sheaf over Y will be denoted F_q and called the *sheaf associated to q*.

We will now proceed to show that this process of associating a sheaf to a function q is indeed the reverse of the process of associating a decomposition to a B-flasque sheaf. Recall from Theorem 3.1.3 that the set of global sections $F_q(Y)$ of the étalé space $p \colon E_q \to Y$ has the structure of a distributive lattice, being a sublattice of the direct product $\prod_{y \in Y} A_y$ of distributive lattices. By Proposition 3.3.1, there is a well-defined function $\eta_q \colon A \to F_q(Y)$ which sends a to $\eta_q(a) := s_a$.

Lemma 3.3.3. *For any continuous map $q \colon X \to Y^\partial$, the function $\eta_q \colon A \to F_q(Y)$ is an injective lattice homomorphism.*

Proof. It is clear from the definition of $F_q(Y)$ that η_q is a homomorphism. We prove that η_q is injective. If $a, b \in A$ and $a \neq b$, then, by Priestley duality, there exists $x \in X$ such that x is in exactly one of $\hat{a}$ and $\hat{b}$. Let $y := q(x)$. Then in particular $x \in X_y$, so that $\hat{a} \cap X_y \neq \hat{b} \cap X_y$, and therefore $f_y(a) \neq f_y(b)$, by definition of f_y. We conclude that $s_a(y) \neq s_b(y)$, so that $\eta_q(a) \neq \eta_q(b)$. $\qquad \square$

The following key definition dually characterizes those continuous maps q which correspond to B-flasque sheaf representations of the distributive lattice A.

Definition 3.3.4. Let X be a Priestley space and Y a stably compact space with lattice basis B. A continuous map $q \colon X \to Y^\partial$ is a *B-patching decomposition of X over Y^∂* if it satisfies the following property for all $U \in B$:

(P_U) If $(U_i)_{i \in I}$ is a collection of open sets in Y, $U = \bigcup_{i \in I} U_i$, and $(D_i)_{i \in I}$ is a collection of clopen downsets in X such that, for all $i, j \in I$,

$$D_i \cap q^{-1}(U_i \cap U_j) = D_j \cap q^{-1}(U_i \cap U_j),$$

then there exists a clopen downset D in X such that
$D \cap q^{-1}(U) = \bigcup_{i \in I} (D_i \cap q^{-1}(U_i))$.

Lemma 3.3.5. *Let $q : X \to Y^{\partial}$ be a continuous map from a Priestley space X to the co-compact dual of a stably compact space Y, and let $p : E_q \to Y$ be the étalé space associated to q. For any open set U of Y, the following statements are equivalent:*

1. *The function q satisfies the property (P_U) in Definition 3.3.4;*

2. *If $s : U \to E_q$ is a continuous section of p over U, then there exists $a \in A$ such that $(s_a)|_U = s$.*

Proof. Let $U \subseteq Y$ be open. For (1) implies (2), suppose that $s : U \to E_q$ is a continuous section over U. For each $y \in U$, we have $s(y) \in A_y$, so by definition we can pick $b(y) \in A$ such that $s(y) = f_y(b(y))$. The set $U_y := s^{-1}(s_{b(y)}(Y))$ is open since s is continuous. Write D_y for the clopen downset $\widehat{b(y)}$ of X. We will now show that $(U_y)_{y \in U}$ and $(D_y)_{y \in U}$ satisfy the assumptions of property (P_U) in Definition 3.3.4. Clearly, $U = \bigcup_{y \in U} U_y$. Note that, for any $z \in U_y$, we have $s(z) = s_y(z) = f_z(b(y))$, but also $s(z) = f_z(b(z))$ by definition of $b(z)$, so $f_z(b(y)) = f_z(b(z))$. Thus, by definition of f_z, D_y and D_z, we get that $D_y \cap X_z = D_z \cap X_z$ for any $z \in U_y$. Let $y, y' \in U$ and let $x \in D_y \cap q^{-1}(U_y \cap U_{y'})$. Writing $z := q(x)$, we get that $x \in D_y \cap X_z = D_z \cap X_z = D_{y'} \cap X_z$, because $z \in U_y$ and $z \in U_{y'}$. We have now proved that $D_y \cap q^{-1}(U_y \cap U_{y'}) \subseteq D_{y'}$, and the other inclusion follows by symmetry. Since q satisfies (P_U), pick a clopen downset D in X such that

$$D \cap q^{-1}(U) = \bigcup_{y \in Y} (D_y \cap q^{-1}(U_y)). \tag{*}$$

Since X is the dual space of A, pick $a \in A$ such that $\widehat{a} = D$. We now show that $(s_a)|_U = s$. Let $y \in U$ be arbitrary. By definition of $b(y)$, we have $s(y) = f_y(b(y))$. We need to show that $f_y(a) = f_y(b(y))$, or, equivalently, that $\widehat{a} \cap X_y = \widehat{b(y)} \cap X_y$. Let $x \in X_y$ be arbitrary. We show that x is in $\widehat{a}$ if, and only if, x is in $\widehat{b(y)}$. By definition of X_y, we have $q(x) \le y \in U_y$, so $q(x) \in U_y$ since U_y is open. Therefore, if $x \in \widehat{a} = D$, we have $x \in D \cap q^{-1}(U)$, and by (*) there is $y' \in Y$ such that $x \in D_{y'} \cap q^{-1}(U_{y'})$. We then also get $x \in D_{y'} \cap q^{-1}(U_y \cap U_{y'}) \subseteq D_y = \widehat{b(y)}$, as we already proved above. Conversely, if $x \in \widehat{b(y)}$, then $x \in D_y \cap q^{-1}(U_y)$, so it follows from (*) that $x \in D \cap q^{-1}(U)$.

For (2) implies (1), let $(U_i)_{i \in I}$ and $(D_i)_{i \in I}$ be as in the assumptions of (P_U). For each $i \in I$, pick $b_i \in A$ such that $\widehat{b_i} = D_i$. Note that, for any $i, j \in I$, if $y \in U_i \cap U_j$, then $X_y = q^{-1}(\downarrow y) \subseteq q^{-1}(U_i \cap U_j)$, and it therefore follows by the assumption of (P_U) that $D_i \cap X_y = D_j \cap X_y$, i.e., $f_y(b_i) = f_y(b_j)$. We define a section $s : U \to E_q$ of p. For $y \in U$, pick $i \in I$ such that $y \in U_i$, and let

$s(y) := f_y(b_i)$. The definition of $s(y)$ does not depend on the choice of i by our remarks above. We show that s is continuous. To this end, let $b \in A$ and $V \subseteq Y$ open, and suppose that $s(y) = s_b(y)$ for some $y \in U \cap V$. Pick $i \in I$ such that $y \in U_i$, so that $s(y) = f_y(b_i)$. The set $W := (s_b)^{-1}(s_{b_i}(Y)) \cap U_i \cap V$ is open in Y since s_b is continuous, and contains y. Note that, for any $z \in W$, we have $s(z) = s_{b_i}(z)$ since $z \in U_i$, and $s_{b_i}(z) = s_b(z)$ since $z \in (s_b)^{-1}(s_{b_i}(Y))$. Hence, $s = s_b$ on an open neighbourhood of y, proving that s is continuous. By (2), pick $a \in A$ such that $s = (s_a)|_U$. It remains to prove that the clopen downset $D := \hat{a}$ satisfies the equality $D \cap q^{-1}(U) = \bigcup_{i \in I}(D_i \cap q^{-1}(U_i))$. If $y \in U_i$, then from the chain of equalities $f_y(a) = s_a(y) = s(y) = f_y(b_i)$ we get that $D \cap X_y = D_i \cap X_y$. The equality now easily follows from the fact that $x \in q^{-1}(U)$ if, and only if, $x \in q^{-1}(U_i)$ for some $i \in I$. $\qquad\square$

Proposition 3.3.6. *Let $q \colon X \to Y^\partial$ be a continuous map from a Priestley space X to the co-compact dual of a stably compact space Y with a lattice basis B, and let F_q be the sheaf associated to q. The following statements are equivalent:*

1. *The map q is a B-patching decomposition of X over Y^∂;*

2. *The sheaf F_q is B-flasque and the map $\eta_q : A \to F_q(Y)$ is an isomorphism.*

Proof. For (1) implies (2), note that in particular q satisfies (P_Y). Therefore, by Lemma 3.3.5, the function η_q is surjective, and thus an isomorphism by Lemma 3.3.3. Also, since each s_a is a continuous global section by Proposition 3.3.1, Lemma 3.3.5 and the fact that q satisfies (P_U) for all $U \in B$ together imply that F_q is B-flasque. For the direction (2) implies (1), assume that F_q is B-flasque and η_q is an isomorphism. Note that q then satisfies condition (2) in Lemma 3.3.5 for any $U \in B$. Therefore, by Lemma 3.3.5, q is a B-patching decomposition. $\qquad\square$

Theorem 3.3.7. *Let A be a distributive lattice with dual Priestley space X, and let Y be a stably compact space with lattice basis B. The B-flasque sheaves over Y with lattice of global sections A are in one-to-one correspondence with the B-patching decompositions of the space X over Y^∂.*

Proof. We prove that the assignments $F \mapsto q_F$ (Definition 3.2.8) and $q \mapsto F_q$ (Definition 3.3.2) are mutually inverse up to isomorphism. By Proposition 3.3.6, if $q \colon X \to Y^\partial$ is a B-patching decomposition, then the associated sheaf F_q is B-flasque and has A as its lattice of global sections, modulo the isomorphism η_q. Let $q' := q_{F_q}$ be the decomposition associated to F_q. By definition of F_q, the stalk of F_q at $y \in Y$ is the lattice A_y dual to $X_y = q^{-1}(\downarrow y)$. It now follows immediately from the definition of q' that $q' = q$, modulo the

homeomorphism $(F_q(Y))_* \cong X$ that is dual to the isomorphism η_q. Conversely, if F is a B-flasque sheaf over Y whose lattice of global sections is A, let $q_F : X \to Y^\partial$ be the decomposition associated to F and let $F' := F_{q_F}$ be the sheaf associated to q_F. By definition, the stalk of the sheaf F' at $y \in Y$ is the lattice dual to $q_F^{-1}(\downarrow y) \subseteq X$, which is exactly the stalk A_y of the sheaf F at y, by Proposition 3.2.9. Hence, the étalé space of F' is isomorphic to the étalé space of F, and therefore F and F' are naturally isomorphic as sheaves by Theorem 3.1.3. In particular, $F' = F_{q_F}$ is B-flasque and η_{q_F} is an isomorphism, so that q_F is a B-patching decomposition by Proposition 3.3.6.   □

The proof of this theorem shows that there is a bijection between the set of isomorphism classes of B-flasque sheaves over Y whose lattice of global sections is isomorphic to A, and the set of isomorphism classes of B-patching decompositions over Y^∂ of spaces homeomorphic to the space X. This bijection can be extended to an equivalence between a category of sheaves of distributive lattices that are flasque on a basis and a category of patching decompositions of Priestley spaces. We leave the precise formulation of this equivalence to future work.

Example 3.3.8. Suppose that A is a distributive lattice which is a direct product of a collection $(A_i)_{i \in I}$ of distributive lattices. We apply our analysis in this chapter to prove that the Priestley space X dual to A has a decomposition over the Boolean space $Y := \beta I$, the Stone-Čech compactification of the set I with the discrete topology. We denote the basis of clopen sets for $Y = \mathcal{P}(I)_*$ by $B = \{\widehat{S} \mid S \subseteq I\}$. There is a sheaf F over Y such that, for each $S \subseteq I$, $F(\widehat{S}) = \prod_{i \in S} A_i$. Indeed, it is easy to see that F is a sheaf on the basis of clopen sets, so that it extends to a sheaf over Y, cf. [108, Theorem II.1.3]. The stalk of F at $y \in Y$ is the ultraproduct A_y of $(A_i)_{i \in I}$ over the ultrafilter y. By Theorem 3.1.3, the direct product $\prod_{i \in I} A_i$ can be identified with the global sections of the sheaf F, that is, the Boolean product of the ultraproducts A_y. We remark in passing that this is the essential observation underlying the well-known Jónsson's Lemma in universal algebra [83], also see [58, Theorem 3.17].

By Remark 3.2.3, the sheaf F is B-flasque. Note that, since Y is a Boolean space, $Y = Y^\partial$ and the specialization order on Y is trivial. Therefore, by Theorem 3.3.7, there is a B-patching decomposition $q_F : A_* \to Y$. We conclude that *the dual space X of a direct product $\prod_{i \in I} A_i$ of distributive lattices decomposes as the disjoint union of closed subspaces X_y, for $y \in \beta I$, where each X_y is the distributive lattice dual to the ultraproduct of $(A_i)_{i \in I}$ over the ultrafilter y.* This fact was proved by different methods in [94, Proposition 3.11] and has recently been applied in the study of forbidden configurations in Priestley spaces, cf. e.g. [5].

We briefly discuss how the results in this chapter can also be applied to distributive lattices with additional operations. If the distributive lattice A is equipped with an additional n-ary operation $f : A^n \to A$ that preserves or reverses joins or meets in each coordinate, then this operation can be represented as an $(n+1)$-ary relation $R \subseteq X \times X^n$ on the Priestley dual space X of A (cf., e.g., [52, Section 2]). Suppose that F is a sheaf of distributive lattices with global sections A. Then F is also a sheaf with respect to the additional operation f if, and only if, each stalk map $A \twoheadrightarrow A_y$ respects the operation f. This condition holds if, and only if, each subspace X_y is a generated subframe with respect to R, that is, for any $x, x_1, \ldots, x_n \in X$, if $x_1, \ldots, x_n \in X_y$ and $xR(x_1, \ldots, x_n)$, then $x \in X_y$.

Concluding remarks

In Theorem 3.3.7, we obtained a one-to-one correspondence between B-patching decompositions and B-flasque sheaves. We believe that there is a more general result which holds for sheaves which are not necessarily B-flasque; we leave the precise statement of this result to further work. Another natural question that we will leave to future work is whether the one-to-one correspondence in Theorem 3.3.7 can be extended to an equivalence between appropriate categories of B-patching decompositions and B-flasque sheaves.

Chapter 4. Sheaf representations of MV-algebras and lattice-ordered abelian groups

We study representations of MV-algebras — equivalently, unital lattice-ordered abelian groups — through the lens of Stone-Priestley duality, using canonical extensions as an essential tool. Specifically, the theory of canonical extensions implies that the (Stone-Priestley) dual spaces of MV-algebras carry the structure of topological partial commutative ordered semigroups. We use this structure to obtain two different decompositions of such spaces, one indexed over the prime MV-spectrum, the other over the maximal MV-spectrum. These decompositions yield sheaf representations of MV-algebras, using the results developed in Chapter 3. Importantly, the proofs of the MV-algebraic representation theorems that we obtain in this way are distinguished from the existing work on this topic by the following features: (1) we use only basic algebraic facts about MV-algebras; (2) we show that the two aforementioned sheaf representations are special cases of a common result, with potential for generalizations; and (3) we show that these results are strongly related to the structure of the Stone-Priestley duals of MV-algebras. In addition, using our analysis of these decompositions, we prove that MV-algebras with isomorphic underlying lattices have homeomorphic maximal MV-spectra. This result is an MV-algebraic generalization of a classical theorem by Kaplansky stating that two compact Hausdorff spaces are homeomorphic if, and only if, the lattices of continuous $[0,1]$-valued functions on the spaces are isomorphic. This chapter is a modified version of the paper [65].

MV-algebras were introduced by C. C. Chang [23] to provide algebraic semantics for Łukasiewicz infinite-valued propositional logic [25], thus playing an analogous role to that of Boolean algebras in classical propositional logic. As proved in [120, Theorem 3.9], MV-algebras are categorically equivalent to unital lattice-ordered abelian groups: the unit interval of any such group forms an MV-algebra, from which the original group and order structure can be naturally recovered; and all MV-algebras arise in this manner. MV-algebras also entertain a wide range of connections with other realms of mathematics, including probability theory, C*-algebras and polyhedral geometry; see [121] for a recent account.

In this chapter we study the *Stone-Priestley dual spaces* of MV-algebras, by which we mean the Stone-Priestley dual spaces of the underlying lattices of MV-algebras equipped with the structure coming from the MV-algebraic

operations. There are several dualities for MV-algebras and lattice-ordered groups (henceforth abbreviated as "ℓ-groups") that generalize Stone duality for Boolean algebras. These results can be roughly divided into two strands. In the first strand one views MV-algebras as groups with a compatible lattice order, in light of the categorical equivalence mentioned above, and looks for representations by continuous real-valued functions in the style of the Stone-Gelfand duality theory for commutative C*-algebras. At the center of this first strand, there is the Baker-Beynon duality between finitely presented abelian ℓ-groups and homogeneous rational polyhedral sets [3, 11]. The affine version of this result shows that finitely presented MV-algebras are the coordinate algebras of rational polyhedra under piecewise linear maps that preserve the arithmetic structure of the polyhedra in an appropriate sense [113, 114]. These theorems highlight the profound relationship between abelian ℓ-groups, MV-algebras, and piecewise linear arithmetic topology; for further recent manifestations of this phenomenon, see e.g. [111, 109, 22, 110, 112, 123]. Dualities for other classes of MV-algebras and ℓ-groups that fall into this first strand can be found in [28, 26, 29].

The second strand of duality results for MV-algebras arises from viewing MV-algebras as distributive lattices with two additional operations $\neg$ and $\oplus$ and investigating the additional structure that these operations yield on the Stone-Priestley dual space of the distributive lattice underlying the MV-algebra. This approach can be traced back several decades in the literature on duality for MV-algebras and ℓ-groups, see e.g. [9, 115, 116, 117, 118, 61, 62]. In particular, [117] establishes a duality theorem between ℓ-groups and a class of spaces dubbed "ℓ-spaces". The more recent papers [61, 62] show that first-order axioms on the Stone-Priestley duals will capture a large class of varieties including MV-algebras, but these papers do not consider MV-algebras specifically. In this chapter, we apply the general insights gained in [61, 62] to the theory of MV-algebras *per se*, providing a systematic approach to sheaf representations of MV-algebras by means of their dual spaces. On the way to these results we obtain an MV-algebraic generalization of a classical theorem by Kaplansky on the lattice of continuous real-valued functions on a compact Hausdorff space [89]. We will now explain our methodology and the ensuing applications in further detail.

A large part of the mathematical machinery used in this paper originates in the theory of canonical extensions. Canonical extensions have been used in the study of modal logic to provide algebraic proofs of the canonicity of modal axioms; see e.g. [59]. Gehrke and Priestley showed in [60] that one of the defining axioms of MV-algebras (specifically, equation (4.1) in Section 4.1 below) is not canonical. The papers [61, 62] studied canonical extensions and duality for double quasi-operators, i.e. operations that pre-

serve ∨ and ∧ in each coordinate, of which the MV-algebraic operations ¬ and ⊕ are prime examples. We rely on the results obtained in [61, 62], with particular focus on their repercussions for MV-algebras. In particular, these results enable us to prove, in Proposition 4.3.7, that the dual Stone-Priestley space of any MV-algebra has the structure of a topological partial commutative ordered semigroup.

In order to study sheaf representations, we use the results from Chapter 3. The setting of stably compact spaces allows us to uniformly treat sheaf representations over both spectral and compact Hausdorff spaces. We show that dual spaces of MV-algebras always admit patching decompositions in the sense of Definition 3.3.4, thus providing a new view of sheaf representations for MV-algebras.

Sheaf representations for ℓ-groups and MV-algebras originate with Keimel [91], [12, Chapitre 10], who established *inter alia* a result for unital abelian ℓ-groups that translates in a standard manner to the following result for MV-algebras: every MV-algebra is isomorphic to the global sections of a sheaf of local MV-algebras on its spectral space of prime MV-ideals equipped with the spectral topology. Here, an MV-algebra is *local* if it has exactly one maximal ideal. From Keimel's work one easily obtains a compact Hausdorff representation, by restricting the base space to the subspace of maximal MV-ideals. In this form, the result was first proved for MV-algebras by Filipoiu and Georgescu [46]; their paper is independent of Keimel's treatment of ℓ-groups. One can also consider the collection of prime MV-ideals equipped with the co-compact dual of the spectral topology. There exists a sheaf representation over this base space whose stalks are totally ordered MV-algebras, which may be regarded as easier to understand than local MV-algebras. The price one pays for this simplification is that the base space is a spectral space which is not T_1 in general, so that it has a non-trivial specialization order. Such a representation for abelian (not necessarily unital) ℓ-groups was established by Yang in his PhD thesis [142, Proposition 5.1.2]; and in [142, Remark 5.3.11] the author remarked that the results translate to MV-algebras via the categorical equivalence mentioned above. In Poveda's PhD thesis [126, Teorema 6.7], the same sheaf representation of MV-algebras via the co-compact dual of the spectral topology on prime MV-ideals was obtained. The interested reader is also referred to the papers [132, 38]. For a further recent proof of the sheaf representation for ℓ-groups via the co-compact dual of the spectral topology, see [133]. For a more thorough historical account of sheaf representations in universal algebra, with particular attention to lattice-ordered groups and rings, see [92] and the references therein.

Apart from giving a unified account of these results on sheaf representations for MV-algebras, we also use our analysis of the dual space of an MV-

algebra to show that MV-algebras with isomorphic underlying lattices have homeomorphic maximal MV-spectra equipped with the spectral topology (Theorem 4.5.5). This result generalizes a theorem by Kaplansky [89] on lattices of continuous functions; see Section 4.5 below for a more detailed comparison.

Outline of the chapter. In Section 4.1 we discuss background on MV-algebras. In Section 4.2, we discuss background on lifting maps and equations to the canonical extension. The analysis of the dual space of an MV-algebra, carried out in Section 4.3, allows us to obtain in Section 4.4 a decomposition of the dual Stone-Priestley space of an MV-algebra indexed by the prime MV-spectrum. In Section 4.5 we prove the generalization of Kaplansky's theorem (Theorem 4.5.5). By combining the results of Chapter 3 and Section 4.4, in Section 4.6 we establish both sheaf representations of an MV-algebra discussed above. As related topics we also discuss the Chinese Remainder Theorem for MV-algebras and the construction of an explicit MV-algebraic term representing the sections occurring in the sheaf representations.

4.1. MV-algebras and their spectra

In this section, we recall the basic definitions of MV-algebras and the different (lattice, prime, maximal) spectra that have been associated to them in the literature. We also describe the relationship between these spectra and the dual spaces of distributive lattices naturally associated to the MV-algebra. We use a minimal amount of MV-algebraic theory in this paper. Indeed, almost all results that we need can be found in [25, Chapter 1]. In particular, we neither assume Chang's completeness theorem [25, 2.5.3], nor even the easier subdirect representation theorem [25, 1.3.3]; the latter is a straightforward consequence of the first sheaf representation given in Section 4.6.

Background references for MV-algebras are [25, 121]. We recall that an *MV-algebra* is an algebraic structure $(M, \oplus, \neg, 0)$, where $0 \in M$ is a constant, $\neg$ is a unary operation satisfying $\neg\neg x = x$, $\oplus$ is a binary operation making $(M, \oplus, 0)$ a commutative monoid, the element 1 defined as $\neg 0$ satisfies $x \oplus 1 = 1$, and the law

$$\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x \tag{4.1}$$

holds. Any MV-algebra has a natural partial order defined by $x \leq y$ if, and only if, there exists z such that $x \oplus z = y$. This partial order is a distributive lattice order bounded below by 0 and above by 1, in which binary suprema are given by $x \vee y = \neg(\neg x \oplus y) \oplus y$. Thus, the characteristic law

(4.1) states that $x \vee y = y \vee x$. Meets can be calculated by the De Morgan equation $x \wedge y = \neg(\neg x \vee \neg y)$. It is common to call *MV-chains* those MV-algebras whose underlying order is total. For $m \geq 1$ an integer, and x an element of an MV-algebra, we often abbreviate by mx the m-fold addition $x \oplus \cdots \oplus x$. We set, as usual, $x \ominus y := \neg(\neg x \oplus y)$. Boolean algebras are precisely those MV-algebras that are idempotent, meaning that $x \oplus x = x$ holds; equivalently, they are the MV-algebras that satisfy the *tertium non datur* law $x \vee \neg x = 1$. For Boolean algebras, the operations $\oplus$ and $\vee$ coincide, and $\ominus$ is the operation of logical difference, $x \wedge \neg y$.

The real unit interval $[0, 1] \subseteq \mathbb{R}$ can be made into an MV-algebra by defining the operations $x \oplus y := \min\{x + y, 1\}$ and $\neg x := 1 - x$; the neutral element is 0. The underlying lattice order of this MV-algebra coincides with the natural order of $[0, 1]$. Thus, in this example, $\oplus$ can be thought of as 'truncated addition', and $x \ominus y$ as 'truncated subtraction', i.e. $x \ominus y = \max\{x - y, 0\}$. The example is generic by *Chang's Completeness Theorem*: The variety of MV-algebras is generated[1] by the standard MV-algebra $[0, 1]$. Chang's original proof is in [23, Lemma 8]; for a textbook treatment, see [25, 2.5.3]. However, as already emphasized above, our results are obtained independently of Chang's theorem.

An *MV-ideal* of an MV-algebra A is a subset $I \subseteq A$ that is a submonoid (i.e. contains 0 and is closed under $\oplus$) and a downset (i.e. contains $b \in A$ whenever it contains $a \in A$ and $b \leq a$). The MV-ideals of A are in natural bijection with the MV-algebra congruences on A [25, 1.2.6], as follows. For an MV-ideal I, two elements $a, b \in A$ are *equal modulo I*, notation $a \equiv b \bmod I$, if both $a \ominus b$ and $b \ominus a$ are in I. We write A/I to denote the quotient of the MV-algebra A modulo the ideal I. An MV-ideal $I \subseteq A$ is *prime* if it is proper (i.e. $I \neq A$), and for each $a, b \in A$ either $(a \ominus b) \in I$, or $(b \ominus a) \in I$.[2] An MV-ideal $I \subseteq A$ is *maximal* if it is proper, and the only MV-ideal that properly contains I is A itself. A standard argument shows that maximal MV-ideals are prime. Each MV-ideal of A is a lattice ideal of A; cf. [121, Proposition 4.13]. Although the converse fails, we have:

Proposition 4.1.1 ([25, 6.1.1])**.** *Let I be an MV-ideal of an MV-algebra A. Then I is a prime ideal of the underlying lattice of A if, and only if, I is a prime MV-ideal.*

Notation. In this chapter, as in the rest of the thesis, whenever we say 'ideal' or 'filter', we mean 'lattice ideal' or 'lattice filter'. Thus, in this terminology, an MV-ideal is an ideal that is moreover closed under $\oplus$.

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[1] Hence the subalgebra $[0, 1] \cap \mathbb{Q}$ also generates the variety of MV-algebras: if an evaluation into $[0, 1]$ makes two terms unequal, then by the continuity of the MV-algebraic operations on $[0, 1]$ there also is an evaluation into $[0, 1] \cap \mathbb{Q}$ that makes those two terms unequal.

[2] The terminology "prime MV-ideal" is consistent with the terminology "prime (lattice) ideal", cf. Proposition 4.1.1 below.

Proposition 4.1.2 ([25, 1.2.3(v)]). *If $f \colon A \twoheadrightarrow B$ is an onto homomorphism of MV-algebras, then the MV-ideal $f^{-1}(0)$ is prime if, and only if, B is totally ordered and non-trivial.*

We write $\langle S \rangle$ to denote the MV-ideal generated by the subset $S \subseteq A$, namely, the intersection of all MV-ideals of A containing S. When $S = \{s\}$ is a singleton we write $\langle s \rangle$ instead of $\langle \{s\} \rangle$, and speak of *principal MV-ideals*.

Proposition 4.1.3. *For any non-empty subset S of an MV-algebra A, we have*

$$\langle S \rangle = \{a \in A \mid a \le s_1 \oplus \cdots \oplus s_k, \text{ for some finite set } \{s_i\}_{i=1}^{k} \subseteq S\}.$$

Furthermore, for any $a, b, c \in A$ we have $\langle a \rangle \vee \langle b \rangle = \langle a \oplus b \rangle = \langle a \vee b \rangle$, and $\langle a \rangle \cap \langle b \rangle = \langle a \wedge b \rangle$. In particular, finitely generated and principal MV-ideals coincide, and the principal MV-ideals of A form a sublattice of the lattice of all MV-ideals of A.

Proof. The first assertion is [25, 1.2.1], and the remaining ones follow from it through a straightforward verification.   □

We now recall a few basic facts about the interaction between the MV-algebraic operations and the order.

Lemma 4.1.4. *In the following, a, b, and c are arbitrary elements of the MV-algebra A.*

1. *The operation $\ominus$ is lower adjoint to $\oplus$, i.e. $a \ominus b \le c \iff a \le b \oplus c$.*

2. *The operation $\oplus$ preserves all existing meets in each coordinate. That is, for any set $B \subseteq A$ such that $\bigwedge B$ exists, $\bigwedge_{b \in B}(b \oplus a)$ also exists, and we have $\bigwedge_{b \in B}(b \oplus a) = (\bigwedge B) \oplus a$. Similarly for the second coordinate. In particular, $\oplus$ is order-preserving in each coordinate.*

3. *The operation $\ominus$ preserves all existing joins in the first coordinate, and reverses all existing joins to meets in the second coordinate. The latter means that for any set $B \subseteq A$ such that $\bigvee B$ exists, $\bigwedge_{b \in B}(a \ominus b)$ exists and we have $a \ominus (\bigvee_{b \in B} b) = \bigwedge_{b \in B}(a \ominus b)$. In particular, $\ominus$ is order-preserving in the first, and order-reversing in the second coordinate.*

4. *The operation $\oplus$ is join-preserving in the second coordinate, i.e. the equation $a \oplus (b \vee c) = (a \oplus b) \vee (a \oplus c)$ holds. Hence, by commutativity, $\oplus$ is join-preserving in each coordinate.*

5. *The operation $\ominus$ is meet-preserving in the first coordinate, i.e. the equation $(a \wedge b) \ominus c = (a \ominus c) \wedge (b \ominus c)$ holds, and meet-reversing in the second coordinate, i.e. $a \ominus (b \wedge c) = (a \ominus b) \vee (a \ominus c)$.*

6. *The equation* $(a \ominus b) \wedge (b \ominus a) = 0$ *holds.*

7. *The equation*[3] $(a \wedge \neg a) \vee (b \vee \neg b) = b \vee \neg b$ *holds.*

8. *The map* $\neg \colon A \to A$ *is an order-reversing bijection that is its own inverse.*

Proof. Item 1 is [25, 1.1.4(iii)]. Items 2–3 are immediate consequences of the adjunction in 1. Items 4–5 are proved as in [25, 1.1.6], *mutatis mutandis*. Item 6 is [25, 1.1.7]. For item 7, we need to show that $a \wedge \neg a \leq b \vee \neg b$. By item 1, we have $a \leq (a \ominus b) \oplus b$ and $\neg a \leq (b \ominus a) \oplus \neg b$, using the obvious equality $\neg a \ominus \neg b = b \ominus a$. Using items 2 and 6, we now get

$$a \wedge \neg a \leq ((a \ominus b) \oplus b) \wedge ((b \ominus a) \oplus \neg b) \leq [(a \ominus b) \wedge (b \ominus a)] \oplus (b \vee \neg b) = b \vee \neg b.$$

Item 8 is [25, 1.1.3 and 1.1.4(i)]. $\qquad \square$

We write $(X, \tau^{\downarrow})$ for the Stone dual space of (the distributive lattice underlying) A. Recall that to any point $x \in X$ there corresponds a unique prime lattice ideal I_x of A, and the specialization order on X corresponds to the inclusion order on ideals (cf. Example 2.1.6 in Chapter 2). We write $Y \subseteq X$ for the subset corresponding to the prime MV-ideals of A, and $Z \subseteq Y$ for the subset corresponding to the maximal MV-ideals of A. The following holds in Y, though not in X:

Proposition 4.1.5 (Cf. [12, 10.1.11]). *If* $y, y' \in Y$, $y \not\leq y'$ *and* $y' \not\leq y$, *then there exist* $u, v \in A$ *such that* $y \in \hat{u}$, $y' \in \hat{v}$ *and* $u \wedge v = 0$.

Proof. Pick $a \in I_{y'} \setminus I_y$ and $b \in I_y \setminus I_{y'}$. Define $u := a \ominus b$ and $v := b \ominus a$. If we had $u \in I_y$, then, since $a \leq (a \ominus b) \oplus b = a \vee b$, we would get $a \in I_y$, contradicting the choice of a. Therefore, $u \notin I_y$, i.e. $y \in \hat{u}$. The proof that $y' \in \hat{v}$ is similar. Since the equation $(a \ominus b) \wedge (b \ominus a) = 0$ holds in any MV-algebra (Lemma 4.1.4.6), we have $u \wedge v = 0$. $\qquad \square$

As a consequence, we now show that the set of prime MV-ideals of A, ordered by inclusion, is a *root system*, i.e., the upset of any one of its elements is totally ordered. This terminology first arose in the context of lattice-ordered groups, cf. e.g. [12].

Corollary 4.1.6. *For each* $y \in Y$, $\uparrow y$ *is totally ordered by* $\leq$. *Moreover, every prime MV-ideal is contained in a unique maximal MV-ideal.*

Proof. This is [25, 1.2.11(ii) and 1.2.12]. To prove the first assertion, let $y \in Y$ and suppose that two distinct elements $y', y'' \in Y$ lie in $\uparrow y$. Suppose further, by way of contradiction, that y' and y'' are incomparable. By Proposition 4.1.5, there are $u \notin I_{y'}$ and $v \notin I_{y''}$ such that $u \wedge v = 0$. But then $u \in I_y$

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[3]This is the De Morgan-Kleene equation, see [4, Ch. XI].

or $v \in I_y$ since I_y is prime, contradicting either $y \le y'$ or $y \le y''$. The second assertion now follows at once by a standard application of Zorn's lemma to the chain $\uparrow y$. $\square$

Remark 4.1.7. Recall [134, Definition 4.3] that a bounded distributive lattice D is *normal* if, whenever $d_1, d_2 \in D$ satisfy $d_1 \vee d_2 = 1$, there exist $c_1, c_2 \in D$ with $c_1 \wedge c_2 = 0$ such that $c_1 \vee d_2 = 1$ and $c_2 \vee d_1 = 1$. The lattice D is normal if, and only if, every prime ideal of D is contained in a unique maximal ideal of D. In this case, the space of maximal ideals with the Stone topology is Hausdorff, and there is a continuous retraction of the Stone dual space of D onto the subspace of maximal ideals [134, Theorem 4.4]. We will show in Proposition 4.1.8 below that the space Y of prime MV-ideals of A is the Stone dual space of the distributive lattice KCon A of principal MV-congruences of A. In light of Corollary 4.1.6, it then follows that the lattice KCon A is normal, and that the map $m : Y \to Z$ which sends a prime MV-ideal to the unique maximal MV-ideal above it is continuous with respect to the topology $\tau^\downarrow$ on Y and Z. The root system property of $(Y, \le)$ (Corollary 4.1.6) corresponds to the fact that KCon A is even *completely normal*; see [27, and references therein]. $\square$

There is a lattice homomorphism $\lambda \colon A \to \text{Con}\, A$ from A to the lattice of MV-congruences on A which sends an element a to the MV-congruence $\lambda(a)$ generated by the pair $(a, 0)$, or, equivalently, to the principal MV-ideal $\langle a \rangle \subseteq A$. (Indeed, λ clearly preserves 0 and 1; it preserves $\wedge$ and $\vee$ by Proposition 4.1.3.) The image of the homomorphism λ is the lattice KCon A of principal (or, equivalently, finitely generated or compact) MV-congruences of A. Hence, writing σ for the kernel of λ, there is an isomorphism of distributive lattices $A/\sigma \cong \text{KCon}\, A$.

Proposition 4.1.8. *For any MV-algebra A, the Priestley dual space of the distributive lattice A/σ is homeomorphic to the closed subspace Y of the dual Priestley space $(X, \tau^p, \le)$ of A. Hence, $(Y, \tau^\downarrow)$ is homeomorphic to the Stone dual space of A/σ.*[4]

Proof. By Proposition 1.1.12, the Priestley dual space of A/σ is homeomorphic to the closed subspace of X defined by

$$S_\sigma = \{x \in X \mid \forall (a, b) \in \sigma \colon (a \in I_x \leftrightarrow b \in I_x)\}.$$

It thus suffices to show that $S_\sigma = Y$. Let $x \in X$. Suppose first that $x \in Y$, i.e. I_x is an MV-ideal. If $(a, b) \in \sigma$ and $a \in I_x$, then b is in the MV-ideal

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[4]There are several places in the literature on MV-algebras and lattice-ordered groups where this result appears under various guises. The reference closest to the spirit of the present paper is [24], where the authors interpret via Priestley duality Belluce's results on the map λ in [9]. Compare also [38, Definition 8.1, Proposition 8.2, and Corollary 8.9].

generated by a, which is contained in I_x. Hence, $b \in I_x$. The proof that $b \in I_x$ implies $a \in I_x$ is symmetric. Hence, $x \in S_\sigma$. Conversely, suppose that $x \in S_\sigma$. We show that I_x is an MV-ideal. If $a, b \in I_x$, then $a \vee b \in I_x$ since I_x is a lattice ideal. By Proposition 4.1.3 we have $\langle a \vee b \rangle = \langle a \oplus b \rangle$, i.e. $(a \vee b, a \oplus b) \in \sigma$. Since $x \in S_\sigma$, we deduce $a \oplus b \in I_x$, as required. The second statement now follows from Theorem 2.1.10 and Example 2.1.11.2. $\square$

Remark 4.1.9. The set of prime MV-ideals, Y, can also be directly topologized using the MV-ideals of A. The most common such topology on Y is known as the *spectral topology* [121, Definition 4.14]: its open sets are the sets of the form $\{y \in Y \mid I \not\subseteq I_y\}$ as I ranges over all MV-ideals of A. Note that these sets are precisely unions of sets of the form $\widehat{a} \cap Y$, where a ranges over A. Hence, this topology is equal to the topology that is induced on Y, viewed as a subspace of the Stone dual space of the distributive lattice underlying A. The further restriction of the spectral topology to the set of maximal MV-ideals, $Z \subseteq Y$, is traditionally called the *hull-kernel topology*. $\square$

Remark 4.1.10. *MV-filters* — upsets containing 1 and closed under $a \odot b := \neg(\neg a \oplus \neg b)$ — are dual to MV-ideals. Because MV-negation is a dual order-automorphism of the underlying lattice of the MV-algebra A, the map $I \mapsto \neg I$, where $\neg I := \{\neg a \mid a \in I\}$, is a bijection between the lattice ideals and filters. It restricts to a bijection between prime or maximal MV-ideals and prime or maximal MV-filters, respectively. Note, however, that the bijection $I \longmapsto I^c$ between the prime filters and the prime ideals of the underlying distributive lattice of A does not restrict to a bijection between prime MV-ideals and prime MV-filters. Indeed, if I is, say, a maximal MV-ideal of A, then simple examples show that I^c is not in general an MV-filter of A (though it is, of course, a prime filter of the underlying lattice). Following tradition, we use MV-ideals rather than MV-filters in the sequel. In particular, we stress that Y denotes the set of points $y \in X$ such that I_y is a prime MV-ideal; this is not the same as the set Y' of points $y \in X$ such that F_y is a prime MV-filter, even though the two are connected via the natural bijection $\neg$ given above. Cf. the notation adopted at the beginning of Section 4.3 below. $\square$

4.2. Lifting operations and inequalities to the canonical extension

We refer to Section 1.2 in Chapter 1 for basic facts about canonical extension. In this section, we collect the relevant facts about the lifting of operations

and inequalities to the canonical extension. For further background we re-
fer the reader to [58, 61].

Recall that, for any distributive lattice D, the assignment

$$\kappa \colon J^\infty(D^\delta) \longrightarrow M^\infty(D^\delta) \tag{4.2}$$

$$x \quad \overset{\kappa}{\longmapsto} \quad \bigvee\{a \in D \mid x \nleq a\}$$

defines an order isomorphism between the posets of completely join-
irreducible and completely meet-irreducible elements of the canonical ex-
tension D^δ [58, Theorem 2.3]. Both of these sets are order-isomorphic to
the poset underlying the Priestley dual space of D (cf. Example 1.2.5 in
Chapter 1). The bijection κ has the useful property

$$\forall u \in D^\delta, x \in J^\infty(D^\delta) \; : \; x \leq u \iff u \nleq \kappa(x). \tag{4.3}$$

The set of filter elements of D^δ will be denoted by $F(D^\delta)$, and the set of ideal
elements will be denoted by $I(D^\delta)$. Note that, under the isomorphisms
$(D^\delta)^n \cong (D^n)^\delta$ and $(D^\delta)^{\mathrm{op}} \cong (D^{\mathrm{op}})^\delta$, we have that $F((D^\delta)^n) \cong (F(D^\delta))^n$
and $F((D^\delta)^{\mathrm{op}}) \cong I(D^\delta)$; order-dual statements hold for ideal elements (cf.
[58, pp. 19–20]).

Let $f \colon D^n \to D$ be an order-preserving n-ary operation on a distributive
lattice D. For a filter element x of $(D^\delta)^n$, let

$$\overline{f}(x) := \bigwedge\{f(a) \mid x \leq a \in D^n\}, \tag{4.4}$$

and for an ideal element y of $(D^\delta)^n$, let

$$\overline{f}(y) := \bigvee\{f(a) \mid y \geq a \in D^n\}. \tag{4.5}$$

Now, for $u \in (D^\delta)^n$, we define

$$f^\sigma(u) := \bigvee\{\overline{f}(x) \mid u \geq x \in F(D^\delta)^n\}, \tag{4.6}$$

$$f^\pi(u) := \bigwedge\{\overline{f}(y) \mid u \leq y \in I(D^\delta)^n\}. \tag{4.7}$$

The operations (4.6–4.7) are called the *σ-extension* and the *π-extension* of f,
respectively.[5] Although both extensions restrict to $\overline{f}$ on filter and ideal ele-
ments of D^δ, they do not necessarily coincide.[6] The above is easily adapted
to operations which are order-reversing in some coordinates. If a function

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[5]For a more general definition of σ- and π-extensions that does not assume monotonicity
of f, see [58, Section 2.4].

[6]The operation $\oplus$ of Chang's non-simple totally ordered MV-algebra, for example, has dis-
tinct σ- and π-extensions; see [60, Proposition 1].

$g: D \times D \to D$ is, say, order-preserving in the first coordinate and order-reversing in the second coordinate, then g acts as an order-preserving operation on $D \times D^{\mathrm{op}}$, so the preceding definitions apply up to the appropriate order flips. It is an important fact that residuated (=adjoint) pairs of operations lift to the canonical extension:[7]

Proposition 4.2.1. *Let D be a distributive lattice, and suppose that $f: D \times D \to D$ and $g: D \times D^{\mathrm{op}} \to D$ are order-preserving operations such that*

$$\forall a, b, c \in D : f(a,b) \leq c \iff a \leq g(c,b).$$

Then:

$$\forall u, v, w \in D^{\delta} : f^{\sigma}(u,v) \leq w \iff u \leq g^{\pi}(w,v).$$

Proof. We only prove that $f^{\sigma}(u,v) \leq w \Rightarrow u \leq g^{\pi}(w,v)$, the other direction being similar. Let us write $E := D \times D \times D^{\mathrm{op}}$, and define operations $p, s, t : E \to D$ by

$$p(a,b,c) := a, \quad s(a,b,c) := g(c,b),$$
$$t(a,b,c) := 0 \text{ if } f(a,b) \leq c, \text{ and } t(a,b,c) := 1 \text{ if } f(a,b) \not\leq c.$$

Observe that, using the assumption, the inequality $p \leq s \vee t$ holds pointwise on E. Therefore, $p^{\sigma} \leq (s \vee t)^{\sigma}$ holds pointwise on E^{δ}. Note that t is order-preserving (as a function from E to D), while s is order-reversing. Hence, by [59, Lemma 5.11], we have $(s \vee t)^{\sigma} \leq s^{\pi} \vee t^{\sigma}$, so $p^{\sigma} \leq s^{\pi} \vee t^{\sigma}$. Now, it is not hard to see from the definitions of the extensions that, for all $u, v, w \in D^{\delta}$, we have $p^{\sigma}(u,v,w) = u$, $s^{\pi}(u,v,w) = g^{\pi}(w,v)$, and $t^{\sigma}(u,v,w) = 0$ if $f^{\sigma}(u,v) \leq w$. In particular, if $f^{\sigma}(u,v) \leq w$, then we get

$$u = p^{\sigma}(u,v,w) \leq (s^{\pi} \vee t^{\sigma})(u,v,w) = g^{\pi}(w,v) \vee 0 = g^{\pi}(w,v). \qquad \square$$

Equally important will be that certain inequalities also lift to the canonical extension. By a *binary dual operator* we mean a binary operation on D which preserves finite meets in both coordinates. The following proposition is a special case of [56, Theorem 4.6], where the result is proved for operations which in each coordinate preserve either finite joins or finite meets.

Proposition 4.2.2. *Let s and t be terms in the language of distributive lattices with an additional binary function symbol f. For any distributive lattice D, and for any binary dual operator f^D on D, if (D, f^D) satisfies the inequality $s \leq t$ then so does $(D^{\delta}, (f^D)^{\pi})$.*

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[7]This was first proved, in the distributive case, by B. Jónsson during early work in 1996 on the paper [58]. Here we give a new proof based on a general method from [59]. See also [51, Proposition 2] for a different proof, in the context of Heyting algebras.

4.3. The structure of the lattice spectrum of an MV-algebra

In this section and the next we examine the structure of the lattice spectrum of an MV-algebra. In particular, we show that the operations $\oplus$ and $\neg$ of an MV-algebra yield dual operations on the lattice spectrum which make it into a topological partial commutative semigroup with an involution (Propositions 4.3.7 and 4.3.1).

Notation. In the remainder of this chapter, we adopt the following conventions.

- A denotes an MV-algebra, and its bounded distributive lattice reduct. It will be clear from the context which is meant.

- X denotes the set of points of the dual space of A. To each point $x \in X$ there is associated a prime ideal I_x, and a prime filter F_x.

- $Y \subseteq X$ denotes the set of points $y \in X$ such that I_y is a prime MV-ideal of A.

- $Z \subseteq Y$ denotes the set of points $z \in Y$ such that I_z is a maximal MV-ideal of A.

- The partial order $\leq$ on X and on its subsets Y and Z is the inclusion order of the corresponding ideals.

- $\tau^{\downarrow}$ is the Stone topology on X, $\tau^{\uparrow}$ is its co-compact dual, and τ^p is the Priestley topology on X. The restrictions of these topologies to Y and Z are denoted by the same symbols. $\qquad\square$

Recall from Remark 4.1.10 that the map which sends a (prime) MV-ideal I to the (prime) MV-filter $\neg I$ is a bijection. Note that, in terms of canonical extensions, this map is the unique extension $(\neg)^{\sigma} = (\neg)^{\pi}$ of the operation $\neg$. For $x \in X$, we define $i \colon X \to X$ by declaring $i(x)$ to be the unique element of X with

$$I_{i(x)} = \neg F_x, \text{ or equivalently, } F_{i(x)} = \neg I_x. \tag{4.8}$$

Observe that the subspace Y is not stable under the operation i: for $y \in Y$, $I_{i(y)}$ is the complement of the prime MV-filter $\neg I_y$, which is a prime ideal, but not necessarily an MV-ideal.

Proposition 4.3.1. *The map* $i \colon X \to X$ *is an order-reversing homeomorphism of the Priestley space* $(X, \tau^p, \leq)$ *which is its own inverse. Moreover, for all* $x \in X$, *either* $x \leq i(x)$ *or* $i(x) \leq x$.

Proof. Both assertions follows at once from Priestley duality, using items (8) and (7) in Lemma 4.1.4, respectively. See also [59, 6.3.4]. □

Next consider the lift $\overline{\oplus}$ of the operation $\oplus$ to ideal elements of A^δ, as in (4.5). We have:

$$\left(\bigvee I\right) \overline{\oplus} \left(\bigvee J\right) = \bigvee \{a \oplus b \mid a \in I, b \in J\} = \bigvee \{c \mid \exists a \in I, b \in J : c \leq a \oplus b\}.$$

Hence we define $\overline{\oplus}$ on ideals $I, J \subseteq A$ as follows:[8]

$$I \overline{\oplus} J := \{c \in A \mid \exists a \in I, b \in J \text{ such that } c \leq a \oplus b\}. \quad (4.9)$$

Similarly, we lift $\ominus$ to filter elements of $A \times A^{\mathrm{op}}$. For F a lattice filter and I a lattice ideal of A, define the lattice filter

$$F \overline{\ominus} I := \{c \in A \mid \exists a \in F, b \in I \text{ such that } c \geq a \ominus b\}. \quad (4.10)$$

Remark 4.3.2. Let us emphasize that the operations $\overline{\oplus}$ and $\overline{\ominus}$ in (4.9) and (4.10) agree with the lifted operations $\overline{\oplus}$ and $\overline{\ominus}$ on filter and ideal elements of the canonical extension, defined in the preceding Section 4.2, upon using the isomorphisms between the filter elements with lattice filters and the ideal elements with lattice ideals [54, Lemma 3.2]. Hence we can apply the methods from Section 4.2 and the papers [61, 62] to $\overline{\oplus}$ and $\overline{\ominus}$, as we will now do. □

Proposition 4.3.3. *Let I and J be ideals of the MV-algebra A, and let F be a filter of A.*

1. *The ideal $I \overline{\oplus} J$ is generated by the elements $a \oplus b$, for $a \in I$ and $b \in J$.*

2. *The operation $\overline{\oplus}$ is commutative, associative, and has the ideal $\{0\}$ as a neutral element.*

3. *The operation $\overline{\oplus}$ is monotone: if $J \subseteq J'$, then $I \overline{\oplus} J \subseteq I \overline{\oplus} J'$.*

4. *The ideal $I \overline{\oplus} J$ contains 1 if, and only if, there exists $a \in I$ such that $\neg a \in J$.*

5. *The ideal I is an MV-ideal if, and only if, $I \overline{\oplus} I \subseteq I$.*

6. *The set $F \overline{\ominus} I$ is a filter, and*

$$F \overline{\ominus} I \subseteq J^c \iff F \subseteq (J \overline{\oplus} I)^c.$$

---

[8]An operation which is essentially the restriction of (4.9) to prime ideals was considered in the context of Wajsberg algebras in [115].

Proof. The second item is a consequence of Proposition 4.2.2, noting that $\oplus$ is a binary dual operator by Lemma 4.1.4.2. Items 1, 3, 4, and 5 are immediate from the definitions. For item 6, apply Proposition 4.2.1 to the operations $f := \ominus$ and $g := \oplus$, which form an adjoint pair by Lemma 4.1.4.1. Then:

$$\left(\bigwedge F\right) \overline{\ominus} \left(\bigvee I\right) \leq \bigvee J \iff \bigwedge F \leq \left(\bigvee J\right) \overline{\oplus} \left(\bigvee I\right).$$

Since $\bigwedge F \leq \bigvee I$ if, and only if, $F \cap I \neq \emptyset$ by the compactness property of canonical extensions, a straightforward rewriting completes the proof. $\square$

The following proposition is central to the paper [61]; cf. Lemma 4.3 and Theorem 4.4 therein.

Proposition 4.3.4. *Let A be an MV-algebra. The following hold:*

1. *For $x \in M^\infty(A^\delta)$, $y \in I(A^\delta)$, we have $x \overline{\oplus} y \in M^\infty(A^\delta) \cup \{1\}$.*

2. *For $j \in J^\infty(A^\delta)$, $x \in M^\infty(A^\delta)$, we have $j \overline{\ominus} x \in J^\infty(A^\delta) \cup \{0\}$.*

Proof. Adopting the terminology of [61], say that an operation $h \colon A \times A \to A$ is a *double operator* if it preserves binary joins and binary meets in each coordinate. By Lemma 4.1.4, $\oplus \colon A \times A \to A$ and $\ominus \colon A \times A^{\mathrm{op}} \to A$ are double operators. For the first item, set $h(a,b) := b \ominus a$. Then $h(a,0) = 0$ for all $a \in A$. Writing $C := A^\delta$, the right upper adjoint of h^σ is the operation $l \colon C \times C \to C$, which sends (u,v) to $v \oplus^\pi u$, by Proposition 4.2.1. By the proof of [61, Theorem 4.4], we get in particular that l maps an element $(y,x) \in F(C^{\mathrm{op}}) \times M^\infty(C) = I(C) \times M^\infty(C)$ into $M^\infty(C) \cup \{1\}$, so that indeed $l(y,x) = x \oplus^\pi y$ is in $M^\infty(A^\delta) \cup \{1\}$, as required. The proof of the second item is dual, and uses that $\oplus$ is a double operator, to which $\ominus$ is a lower adjoint (Lemma 4.1.4). $\square$

Corollary 4.3.5. *Let $x \in X$ and let J be an ideal of A. Then $I_x \overline{\oplus} J$ is a prime ideal of A if, and only if, $I_{i(x)} \supseteq J$. In particular, given $x, y \in X$, the ideal $I_x \overline{\oplus} I_y$ of A is prime if, and only if, $i(x) \geq y$.*

Proof. By Proposition 4.3.4, if $1 \notin I_x \overline{\oplus} J$, then $I_x \overline{\oplus} J$ is a prime ideal. By Proposition 4.3.3.4, we have $1 \notin I_x \overline{\oplus} J$ if, and only if, there is no $a \in I_x$ such that $\neg a \in J$ if, and only if, $\neg I_x \subseteq J^c$. This is equivalent to $I_{i(x)} \supseteq J$ by the definition (4.8) of $i(x)$. $\square$

We now define a partial binary operation $+$ on X with domain

$$\mathrm{dom}(+) := \{(x,y) \in X^2 \mid i(x) \geq y\} = \{(x,y) \in X^2 \mid 1 \notin I_x \overline{\oplus} I_y\}.$$

For $(x,y) \in \mathrm{dom}(+)$, we let $x + y$ be the unique element of X such that

$$I_{x+y} = I_x \overline{\oplus} I_y. \tag{4.11}$$

Remark 4.3.6. We observe in passing that the dual $i\colon X \to X$ of negation as in (4.8) is definable from $+\colon \mathrm{dom}(+) \to X$. Indeed, one has

$$i(x) = \max\{y \in X \mid (x,y) \in \mathrm{dom}(+)\}.$$

Strictly speaking, therefore, carrying along the structure $i\colon X \to X$ on the dual space is not necessary.

Importantly, we obtain:

Proposition 4.3.7 (Dual structure). *The structure $(X, \tau^\uparrow, +)$ is a topological partial commutative semigroup that is translation-invariant with respect to the specialization order, and whose set of idempotent elements is the MV-spectrum Y of A. More precisely, for all $x, x', x'' \in X$, the following hold.*

1. *(Commutativity.) If $x + x'$ is defined, then $x' + x$ is defined, and $x + x' = x' + x$.*

2. *(Associativity.) If $x + x'$ and $(x + x') + x''$ are defined, then $x + (x' + x'')$ is defined, and $(x + x') + x'' = x + (x' + x'')$.*

3. *(Translation-invariance.) If $x' \leq x''$ and $x + x''$ is defined, then $x + x'$ is defined, and $x + x' \leq x + x''$.*

4. *(Idempotents are Priestley-closed.) The ideal I_x is a prime MV-ideal if, and only if, $x + x$ is defined and $x + x = x$. Hence*

$$Y = \{y \in X \mid (y,y) \in \mathrm{dom}(+) \text{ and } y + y \leq y\}$$
$$= \{y \in X \mid (y,y) \in \mathrm{dom}(+) \text{ and } y + y = y\},$$

and Y is closed in (X, τ^p).

5. *(Continuity.) The function $+ \colon \mathrm{dom}(+) \to X$ is continuous with respect to the topology $\tau^\uparrow$ on X and the topology that $\mathrm{dom}(+)$ inherits from $(X, \tau^\uparrow) \times (X, \tau^\uparrow)$.*

6. *(Closed domain.) The set $\mathrm{dom}(+) \subseteq X^2$ is closed in $(X, \tau^\uparrow) \times (X, \tau^\uparrow)$.*

Proof. The first three items follow directly from Proposition 4.3.3. For item 4, suppose I_x is a prime MV-ideal. Then $I_x \overline{\oplus} I_x \subseteq I_x$ by Proposition 4.3.3.5, so in particular $x + x$ is defined. It follows that $x + x \leq x$. On the other hand, $x \leq x + x$ always holds since $I_x = I_x \overline{\oplus} 0 \subseteq I_x \overline{\oplus} I_x$ by Proposition 4.3.3. Thus we infer $x + x = x$. The converse direction is clear from Proposition 4.3.3.5. The fact that Y is closed in (X, τ^p) was proved in Proposition 4.1.8. For item 5, pick $a \in A$. Note that, for $(x, x') \in \mathrm{dom}(+)$, we have $a \in I_{x+x'}$ if, and only if, there exist $b \in I_x$ and $c \in I_{x'}$ such that $a \leq b \oplus c$, so

$$+^{-1}(\widehat{a}^c) = \mathrm{dom}(+) \cap \bigcup \{\widehat{b}^c \times \widehat{c}^c \mid b, c \in A : a \leq b \oplus c\},$$

which is clearly open in $\mathrm{dom}(+)$. For the last item, note that $\leq$ is a closed subset of $(X, \tau^{\downarrow}) \times (X, \tau^{\uparrow})$, because $(X, \tau^p, \leq)$ is a Priestley space. Since i is an order-reversing continuous function by Proposition 4.3.1, $\mathrm{dom}(+)$ is equal to $\{(x, y) \mid i(x) \geq y\}$, which is closed in $(X, \tau^{\uparrow}) \times (X, \tau^{\uparrow})$. $\qquad\square$

Notation. Henceforth, given $x, x', x'' \in X$, we write $x + x' = x'', x' \leq x' + x''$, and so forth, to mean 'both sides of the (in)equality are defined, and the (in)equality holds'.

4.4. The decomposition of the lattice spectrum: the map k

In this section we construct the map $k \colon Y \to X$ that is central to our results, and establish some of its order-topological properties, which naturally arise here (and, implicitly, in [61]) from the canonical extension of an MV-algebra. In particular, we obtain a decomposition of X into simple fibres in Proposition 4.4.5, and we prove an 'interpolation lemma', Lemma 4.4.9, that will be crucial in our applications.

As a particular instance of Priestley duality, MV-algebraic quotients of A correspond to closed subspaces of the Priestley dual space X. The operation $\overline{\oplus}$ may be used to characterize the closed subspace dual to such a quotient, as follows.

Proposition 4.4.1. *Let J be an MV-ideal. The lattice spectrum of the quotient MV-algebra A/J is homeomorphic to the Priestley-closed subspace of X defined by*

$$S_J := \{x \in X \mid I_x \,\overline{\oplus}\, J \subseteq I_x\}.$$

Moreover, if $J = I_y$ for some $y \in Y$, then $S_{I_y} = \{x \in X \mid x + y \leq x\}$ is totally ordered.

Proof. Let us write ϑ_J for the MV-congruence on A which is the kernel of $A \twoheadrightarrow A/J$; recall that $(a, b) \in \vartheta_J$ if, and only if, $(a \ominus b) \vee (b \ominus a) \in J$. Since A/J is in particular a lattice quotient of A, we may apply Proposition 1.1.12: the lattice spectrum of A/J is homeomorphic to the closed subspace

$$S_J = \{x \in X \mid \forall (a, b) \in \vartheta_J : (a \in I_x \leftrightarrow b \in I_x)\}$$

with the subspace topology and the restricted order. A short argument, using the definition of ϑ_J, shows that $x \in S_J$ if, and only if, for all (a, b) such that $a \ominus b \in J$, $b \in I_x$ implies $a \in I_x$. Rewriting this condition using the definition of $\overline{\ominus}$ (4.10), it is equivalent to $F_x \,\overline{\ominus}\, I_x \subseteq J^c$. This, in turn, is equivalent to $F_x \subseteq (J \,\overline{\oplus}\, I_x)^c$, by Proposition 4.3.3.6. From this chain of equivalences, we conclude that $x \in S_J$ if, and only if, $I_x \,\overline{\oplus}\, J \subseteq I_x$. To prove the second assertion, recall from Proposition 4.1.2 that if $J = I_y$ is prime,

then A/I_y is totally ordered. Hence, S_{I_y} is also totally ordered, because by a standard exercise the prime ideals of a chain form a chain. The fact that $S_{I_y} = \{x \in X \mid x + y \le x\}$ is immediate from the definition of $+$. ☐

Notation. Henceforth, if $y \in Y$, we write $C_y = \{x \in X \mid x + y \le x\}$ for the chain $S_{I_y} \subseteq X$.

Our next aim is to show that, for a fixed $x \in X$, there exists a *maximum* element $y \in Y$ such that $x + y \le x$; this will lead us to the definition of a map $k \colon X \to Y$ (Definition 4.4.4). We prove this through canonical extensions, using the bijection $\kappa \colon J^\infty(A^\delta) \to M^\infty(A^\delta)$ defined in (4.2).

Lemma 4.4.2. *For any $x \in M^\infty(A^\delta)$, the set*

$$\{u \in A^\delta \mid x \oplus^\pi u \le x\}$$

is closed under $\oplus^\pi$, and has a maximum, which is given by $\kappa(\kappa^{-1}(x) \overline{\ominus} x) \in M^\infty(A^\delta)$.

Proof. To see that the set is closed under $\oplus^\pi$, notice that if u, v are in the set, then

$$x \oplus^\pi (u \oplus^\pi v) = (x \oplus^\pi u) \oplus^\pi v \le x \oplus^\pi v \le x,$$

using that $\oplus^\pi$ is associative, by Proposition 4.2.2. Let $j := \kappa^{-1}(x) \in J^\infty(A^\delta)$. By Proposition 4.3.4.2, $j \overline{\ominus} x \in J^\infty(A^\delta) \cup \{0\}$. To obtain a contradiction, suppose that $j \overline{\ominus} x = 0$ in A^δ. Then, by adjunction (Lemma 4.1.4 and Proposition 4.2.1), we would conclude $j \le x \overline{\oplus} 0 = x = \kappa(j)$, which contradicts the definition of $\kappa(j)$. So $j \overline{\ominus} x \in J^\infty(A^\delta)$. Note that, for any $u \in A^\delta$,

$$
\begin{aligned}
x \oplus^\pi u \le x &\iff j \not\le x \oplus^\pi u && \text{(using (4.3) and } x = \kappa(j)) \\
&\iff j \overline{\ominus} x \not\le u && \text{(Proposition 4.2.1)} \\
&\iff u \le \kappa(j \overline{\ominus} x) && \text{(using (4.3)).}
\end{aligned}
$$

Therefore, the maximum of the set $\{u \in A^\delta \mid x \oplus^\pi u \le x\}$ is $\kappa(j \overline{\ominus} x)$, which is indeed an element of $M^\infty(A^\delta)$. ☐

Proposition 4.4.3. (i) *For any $x \in X$, there exists a largest ideal J of A such that $I_x \overline{\oplus} J \subseteq I_x$, namely, the prime ideal $(F_x \overline{\ominus} I_x)^c$.* (ii) *For any $x \in X$, $(F_x \overline{\ominus} I_x)^c$ is in fact a prime MV-ideal.*

Proof. (i) Write $k(x) := \kappa(\kappa^{-1}(x) \overline{\ominus} x)$. By Lemma 4.4.2, the largest such ideal is $I_{k(x)}$, which is indeed equal to $(F_x \overline{\ominus} I_x)^c$. (ii) By Proposition 4.4.2, the set of elements u such that $x \oplus^\pi u \le x$ is closed under $\oplus^\pi$. In particular, $k(x) + k(x)$ is in this set, so $k(x) + k(x) \le k(x)$. It now follows that $I_{k(x)}$ is an MV-ideal by Propostion 4.3.7. ☐

Definition 4.4.4. In light of Proposition 4.4.3, we define a function $k\colon X \to Y$ by letting $k(x) \in Y$ be such that $I_{k(x)}$ is the largest prime MV-ideal of A satisfying $I_x \oplus J \subseteq I_x$; equivalently, $k(x) := \kappa(\kappa^{-1}(x) \overline{\ominus} x)$.

The fibres of the map $k\colon X \to Y$ decompose X in a simple manner.

Proposition 4.4.5 (Decomposition by k). *For any $y \in Y$, we have $k^{-1}(\uparrow y) = C_y$.*

Proof. Let $y \in Y$. For any $x \in X$, we have $k(x) \geq y$ if, and only if, $x + y \leq x$ if, and only if, $x \in C_y$, by definition of C_y. $\qquad\square$

Proposition 4.4.6. *The map $k : X \to Y$ has the following topological properties.*

1. *For $x \in X$, we have*

$$I_{k(x)} = \{a \in A \mid \forall c \in A : (c \ominus a \in I_x \to c \in I_x)\}.$$

2. *The set $k^{-1}(\widehat{a})$ is open in (X, τ^p) for any $a \in A$.*

3. *The map $k\colon (X, \tau^p) \to (Y, \tau^{\downarrow})$ is a continuous function.*

Proof. For (1), recall that $k(x) = \kappa(\kappa^{-1}(x) \overline{\ominus} x)$, when x is regarded as an element of $M^{\infty}(A^{\delta})$. Therefore,

$$
\begin{aligned}
a \notin I_{k(x)} &\iff a \not\leq \kappa(\kappa^{-1}(x) \overline{\ominus} x) \\
&\iff \kappa^{-1}(x) \overline{\ominus} x \leq a && \text{(using (4.3))} \\
&\iff \kappa^{-1}(x) \overline{\ominus} a \leq x && (*) \\
&\iff \bigwedge_{\kappa^{-1}(x) \leq c} (c \ominus a) \leq x && \text{(def. of } \overline{\ominus} \text{ on } F(A^{\delta})) \\
&\iff \exists c \in A : \kappa^{-1}(x) \leq c \text{ and } c \ominus a \leq x && (x \in M^{\infty}) \\
&\iff \exists c \in A : c \notin I_x \text{ and } c \ominus a \in I_x,
\end{aligned}
$$

where the equivalence marked (*) follows easily from the adjunction between $\overline{\ominus}$ and $\overline{\oplus}$ (Lemma 4.1.4.1 and Proposition 4.2.1), and the fact that $\overline{\oplus}$ is commutative. From the above chain of equivalences, (1) is clear. For (2), note that (1) says that $a \in I_{k(x)}$ if, and only if, $x \in \bigcap_{c \in A} (\widehat{c \ominus a} \cup \widehat{c}^c)$. Therefore,

$$k^{-1}(\widehat{a}^c) = \bigcap_{c \in A} (\widehat{c \ominus a} \cup \widehat{c}^c),$$

which is closed in (X, τ^p). (3) is immediate from the definition of the topology $\tau^{\downarrow}$. $\qquad\square$

Remark 4.4.7. Proposition 4.4.6.1 shows that our function k is the same as the function K defined in [25, Theorem 6.1.3], where it is attributed to the PhD thesis of N. G. Martínez, also see [117]. Moreover, by Proposition 4.4.6.2, the set $k^{-1}(\hat{a}^c)$ is closed for any $a \in A$. By Proposition 1.1.12, this set corresponds to a lattice quotient $A \twoheadrightarrow A_a$ of A. A straight-forward argument using the definition of k shows that this quotient $A \twoheadrightarrow A_a$ coincides with the MV-algebraic quotient of A by the principal MV-ideal $\langle a \rangle$.

$\square$

We now prove that the fixed points of the map k are exactly the points of X corresponding to prime MV-ideals. That is, k is a retract of X onto Y.

Proposition 4.4.8. *Let $x \in X$. Then $x \in Y$ if, and only if, $k(x) = x$.*

Proof. For the non-trivial implication, recall from Proposition 4.3.7.4 that if $x \in Y$, then $x + x$ is defined and $x + x = x$. In particular, by definition of $k(x)$, we have $x \leq k(x)$. On the other hand, using translation-invariance of $+$ (Proposition 4.3.7.3), we get that $k(x) = 0 + k(x) \leq x + k(x) \leq x$. We conclude that $x = k(x)$. $\square$

Finally, we show how the map k relates to the order of the space X. This will be of crucial importance in our applications.

Lemma 4.4.9 (Interpolation Lemma). *Let $x, x' \in X$ be such that $x \leq x'$. There exists $x'' \in X$ such that $x \leq x'' \leq x', k(x'') \geq k(x)$ and $k(x'') \geq k(x')$.*

Proof. We use the properties of $+$ established in Proposition 4.3.7. Define $x'' := x + k(x')$. Note that x'' is well-defined, because $k(x') \leq i(x') \leq i(x)$. Clearly, $x \leq x''$, by monotonicity of $+$. Also, $x'' = x + k(x') \leq x' + k(x') \leq x'$. To show that $k(x) \leq k(x'')$, it suffices to show that $x'' + k(x) = x''$, by definition of $k(x'')$. We calculate:

$$x'' + k(x) = x + k(x') + k(x)$$
$$= x + k(x) + k(x')$$
$$= x + k(x') = x''.$$

Similarly, to show that $k(x') \leq k(x'')$, we prove that $x'' + k(x') = x''$:

$$x'' + k(x') = x + k(x') + k(x')$$
$$= x + k(x') = x''.$$

$\square$

4.5. Kaplansky's Theorem for MV-algebras: the map m

Kaplansky [89] proved that for any compact Hausdorff space S the distributive lattice order on the set of real-valued continuous functions, $C(S)$,

uniquely determines the space S (up to homeomorphism). He also re-
marked on how to obtain the analogous result for the lattice of continu-
ous functions with co-domain the unit interval $[0,1] \subseteq \mathbb{R}$. Kaplansky's
result should be compared with the standard Stone-Gelfand-Kolmogorov
theorem that the unital commutative ring structure of $C(S)$ determines S;
see e.g. [68, Theorem 4.9]. The set $C(S,[0,1])$ of all $[0,1]$-valued contin-
uous functions on S is naturally an MV-algebra, with operations defined
pointwise from the standard MV-algebra $[0,1]$. As an application of The-
orem 4.5.5 below, we prove (Corollary 4.5.6) that the underlying lattice or-
der of any separating MV-subalgebra of $C(S,[0,1])$ uniquely determines the
space S.

By Corollary 4.1.6, there is a uniquely determined function

$$m \colon Y \to Z \tag{4.12}$$

that sends each prime MV-ideal to the unique maximal MV-ideal that con-
tains it. By Remark 4.1.7, m is continuous with respect to the topologies $\tau^{\downarrow}$
on Y and Z. We now study the composite function $m \circ k \colon X \to Z$. Define a
binary relation W on X by setting

$$x_1 W x_2 \iff \text{there exist } x_1', x_2', x_0 \in X \text{ such that } x_1' \leq x_1, \, x_2' \leq x_2, \, x_0 \geq x_1', x_2'. \tag{4.13}$$

The picture below depicts the typical order configuration for which $x_1 W x_2$.
The reason for our choice of the notation 'W' for this relation should be clear
from the picture.
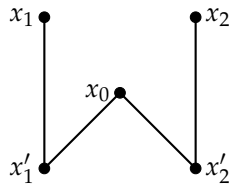
Lemma 4.5.1. *Let $x, x' \in X$ be such that $x \leq x'$. Then $m(k(x)) = m(k(x'))$.*

Proof. By the interpolation lemma (Lemma 4.4.9), there is $y := k(x'') \in Y$
such that $y \geq k(x)$ and $y \geq k(x')$. The maximal MV-ideal above y is above
both $k(x)$ and $k(x')$. Hence, $m(k(x)) = m(k(x'))$, as there is a unique maxi-
mal ideal above $k(x)$ and $k(x')$.   □

Lemma 4.5.2. *For any $x_1, x_2 \in X$, $x_1 W x_2$ if, and only if, $m(k(x_1)) = m(k(x_2))$.
In other words, $W = \ker(m \circ k)$.*

Proof. Suppose $x_1 W x_2$. Pick $x_1', x_2', x_0 \in X$ as in the definition of W. Repeated applications of Lemma 4.5.1 yield

$$m(k(x_1)) = m(k(x_1')) = m(k(x_0)) = m(k(x_2')) = m(k(x_2)).$$

Conversely, suppose $m(k(x_1)) = m(k(x_2))$. Then $x_i \geq k(x_i) =: x_i'$, and we also have $x_0 := m(k(x_1)) = m(k(x_2)) \geq k(x_i)$ for $i = 1, 2$. Hence, $x_1 W x_2$. □

Let us write $[x]_W$ for the equivalence class of $x \in X$ under W, X/W for the quotient set, and

$$q \colon X \longrightarrow X/W$$

for the natural quotient map $x \mapsto [x]_W$. Since $W = \ker(m \circ k)$ by Lemma 4.5.2, there is a unique (set-theoretic) function $f \colon X/W \to Z$ making the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ q\ } & X/W \\
{\scriptstyle m \circ k}\big\downarrow & \raisebox{-1ex}{$\diagup$}{\scriptstyle f} & \\
Z & &
\end{array}
\tag{4.14}
$$

commute, and this function f is a bijection.

Lemma 4.5.3. *The map $m \circ k \colon (X, \tau^{\downarrow}) \to (Z, \tau^{\downarrow})$ is continuous.*

Proof. By Proposition 4.4.6, $k \colon (X, \tau^p) \to (Y, \tau^{\downarrow})$ is continuous, and by Remark 4.1.7, $m \colon (Y, \tau^{\downarrow}) \to (Z, \tau^{\downarrow})$ is continuous, so $m \circ k \colon (X, \tau^p) \to (Z, \tau^{\downarrow})$ is continuous. Now note that, by Lemma 4.5.1, for any subset $T \subseteq Z$, $(m \circ k)^{-1}(T)$ is both an upset and a downset in $(X, \leq)$. In particular, for any open set U in $(Z, \tau^{\downarrow})$, we get that $(m \circ k)^{-1}(U)$ is an open downset in $(X, \tau^p, \leq)$, and therefore it is open in $(X, \tau^{\downarrow})$.[9] □

Proposition 4.5.4. *Let σ denote the quotient topology that X/W inherits from $(X, \tau^{\downarrow})$. Then the unique function $f \colon (X/W, \sigma) \to (Z, \tau^{\downarrow})$ making (4.14) commute is a homeomorphism.*

Proof. Note that f is continuous by Lemma 4.5.3, the fact that (4.14) commutes, and the definition of the quotient topology. As $q \colon (X, \tau^{\downarrow}) \to (X/W, \sigma)$ is continuous and onto, the space $(X/W, \sigma)$ is compact because $(X, \tau^{\downarrow})$ is. Recall that $(Z, \tau^{\downarrow})$ is a Hausdorff space. So f is a continuous bijection from a compact space to a Hausdorff space, and therefore f is a homeomorphism. □

---

[9]Note that $(m \circ k)^{-1}(U)$ is also open in $\tau^{\uparrow}$, so that the function $m \circ k$ remains continuous when we put the stronger topology $\tau^{\downarrow} \cap \tau^{\uparrow}$ on X, but we will not need this in what follows.

We are ready to establish the MV-algebraic generalization of Kaplansky's Theorem.

Theorem 4.5.5 (MV-algebraic Kaplansky's Theorem). *MV-algebras with isomorphic underlying lattices have homeomorphic maximal MV-spectra, i.e. collections of maximal MV-ideals equipped with the spectral topology $\tau^{\downarrow}$.*

Proof. It suffices to show that we can recover $(Z, \tau^{\downarrow})$ from $(X, \tau^{\downarrow})$ only, up to homeomorphism. This is precisely the content of Proposition 4.5.4, upon noting that (4.13) defines the relation W in terms of the specialization order of $(X, \tau^{\downarrow})$. □

For any compact Hausdorff space S, let $C(S, [0,1])$ denote the MV-algebra of all continuous $[0,1]$-valued functions on S, where $[0,1]$ is endowed with its standard (Euclidean) topology, and the operations are defined pointwise from the standard MV-algebra $[0,1]$. A subset $B \subseteq C(S, [0,1])$ is said to *separate the points of S* if for each $p \neq q \in S$ there is $b \in B$ with $b(p) = 0$, $b(q) > 0$. We can now obtain (a stronger version of) [89, Theorem 1 and §6].

Corollary 4.5.6 (Kaplansky's Theorem). *Let S_1, S_2 be any two compact Hausdorff spaces.* (i) *Suppose $A_i \subseteq C(S_i, [0,1])$ is an MV-subalgebra that separates the points of S_i, $i = 1, 2$. If A_1 and A_2 are isomorphic as lattices, then S_1 and S_2 are homeomorphic.* (ii) *The lattices $C(S_1, [0,1])$ and $C(S_2, [0,1])$ are isomorphic if, and only if, the spaces S_1 and S_2 are homeomorphic.*

Proof. (i) Let Z_i denote the maximal MV-spectrum of A_i, $i = 1, 2$, equipped with the spectral topology $\tau^{\downarrow}$. Since A_i separates the points of S_i, by [25, 3.4.3] it follows that S_i is homeomorphic to Z_i, $i = 1, 2$. By assumption, the lattice reducts of A_1 and A_2 are isomorphic. By Theorem 4.5.5, Z_1 and Z_2 are homeomorphic. So S_1 is homeomorphic to S_2. (ii) For the non-trivial implication, write A_i for the MV-algebra $C(S_i, [0,1])$, $i = 1, 2$. Now S_i is compact Hausdorff by hypothesis, and therefore by Urysohn's Lemma the elements of A_i separate the points of S_i. By [25, 3.4.3] it follows that S_i is homeomorphic to the maximal MV-spectrum of A_i ($i = 1, 2$) with the spectral topology. Apply (i). □

Theorem 4.5.5 does not extend without substantial modifications to prime MV-spectra, as the following example shows.

Example 4.5.7. For readers familiar with ℓ-groups, the MV-algebra that we will consider in this example is the unit interval of $(\mathbb{Q}\vec{\times}\mathbb{Q}, (1,1))$, where $\mathbb{Q}\vec{\times}\mathbb{Q}$ denotes, as usual, the lexicographic product of the ordered additive abelian group $\mathbb{Q}$ of rational numbers with itself.
Consider the subalgebra $Q = [0,1] \cap \mathbb{Q}$ of the standard MV-algebra $[0,1]$. Totally order the set-theoretic Cartesian product $\mathbb{Q} \times \mathbb{Q}$ lexicographically.

That is, define $(a, b) \preceq (a', b')$, for $a, b, a', b' \in Q$, to mean $a < a'$, or else $a = a'$, and then $b \leq b'$. Next, define the operation $\oplus$ on $Q \times Q$ by setting $(a, b) \oplus (a', b') := \min\{(a + a', b + b'), (1, 1)\}$, where the minimum is taken with respect to $\preceq$. Further define a unary operation $\neg$ on Q by $\neg(a, b) := (1 - a, 1 - b)$. It is straightforward to check that $Q \times Q$ is an MV-algebra under the binary operation $\oplus$, the unary operation $\neg$, and the constant $(0, 0)$. A further verification shows that the underlying order of this MV-algebra coincides with the restriction of $\preceq$ to $Q \times Q$. Hence $Q \times Q$ is a totally ordered MV-algebra whose underlying lattice is a dense countable chain with endpoints. But any two such chains are order-isomorphic [131, 2.9], and so the MV-algebras $Q \times Q$ and Q have isomorphic underlying lattices. Now direct inspection shows that Q is non-trivial and simple (=it has no non-trivial congruences) and therefore its prime MV-spectrum is the singleton $\{0\}$; whereas $Q \times Q$ has a unique non-maximal prime MV-ideal — namely, the MV-ideal $\{(0, 0)\}$, which lies below the unique maximal MV-ideal $\{(0, q) \mid q \in Q\}$ — whence its MV-spectrum is a doubleton. $\square$

4.6. Sheaf representations from k and m

We now combine the theory developed in Chapter 3 with the specific information about dual spaces of MV-algebras obtained in Section 4.4. This leads to the two known sheaf representations of MV-algebras discussed in the introduction to this chapter; one over the space of prime MV-ideals, the other over the space of maximal MV-ideals. We will first use the map k, defined in Definition 4.4.4. By Proposition 3.3.6, it suffices to show that k is a B-patching decomposition for some basis B. We now exhibit such a basis.

Proposition 4.6.1. *Let B be the basis $\{\hat{a}^c \cap Y \mid a \in A\}$ for $(Y, \tau^\uparrow)$. The function $k \colon X \to Y$ is a B-patching decomposition over $(Y, \tau^\downarrow) = (Y, \tau^\uparrow)^\partial$.*

Proof. Recall from Section 2.1 in Chapter 2 that the space $(Y, \tau^\downarrow)$ is indeed the co-compact dual of $(Y, \tau^\uparrow)$. The map $k : X \to Y$ is continuous from (X, τ^p) to $(Y, \tau^\downarrow)$ by Proposition 4.4.6. It remains to prove that k satisfies the property (P_U) in Definition 3.3.4 for any $U \in B$. Notice that, since $(Y, \tau^\uparrow)$ is a spectral space and the sets in B are compact, it suffices to consider finite covers by compact-open sets. Also observe that, for any $U \in B$, the set $k^{-1}(U)$ is closed in X by Proposition 4.4.6.2. By Priestley duality, a clopen downset of a closed subspace C of X can always be written as the intersection of C and a clopen downset of X (cf. Prop. 1.1.12 or [36, Exercise 11.12(i)]). Combining these observations, in this case it suffices to prove the following property for any $U \in B$.

(P'_U) Suppose that $(U_i)_{i=1}^n$ is a finite collection of compact-open sets in

$(Y, \tau^\uparrow)$ such that $U = \bigcup_{i=1}^n U_i$, and that $(D_i)_{i=1}^n$ is a collection of clopen downsets in X such that, for all $i, j \in \{1, \ldots, n\}$,

$$D_i \cap k^{-1}(U_i \cap U_j) = D_j \cap k^{-1}(U_i \cap U_j).$$

Then the set $D := \bigcup_{i=1}^n (D_i \cap k^{-1}(U_i))$ is a clopen downset in the subspace $k^{-1}(U)$.

Let $(U_i)_{i=1}^n$ and $(D_i)_{i=1}^n$ be finite collections as in (P_U'). We need to show that the set $D := \bigcup_{i=1}^n (D_i \cap k^{-1}(U_i))$ is a clopen downset in the subspace $k^{-1}(U)$. By Proposition 4.4.6, $k^{-1}(U_i)$ is closed for each i. Since each D_i is closed, it now follows that D is closed. To show that D is open in $k^{-1}(U)$, we will prove that

$$k^{-1}(U) \setminus D = \bigcup_{i=1}^n ((D_i)^c \cap k^{-1}(U_i)), \qquad (\star)$$

where the right-hand-side in $(\star)$ is again clearly closed in X. To prove $(\star)$, notice that the assumption on the sets D_i implies that, if x is in $D_i \cap k^{-1}(U_i)$ for *some* i, then $x \in D_j$ for *all* j such that $x \in k^{-1}(U_j)$. The equality $(\star)$ now easily follows from the fact that the sets $k^{-1}(U_i)$ cover $k^{-1}(U)$. Finally, we show that D is a downset in $k^{-1}(U)$. Suppose that $x' \in K$ and $x \leq x'$ for some $x \in k^{-1}(U)$. We prove that $x \in D$. Since $x' \in D$, pick $i \in \{1, \ldots, n\}$ such that $x' \in D_i \cap k^{-1}(U_i)$, and since the U_i's cover U, pick $j \in \{1, \ldots, n\}$ such that $k(x) \in U_j$. By the interpolation lemma (Lemma 4.4.9), pick $x'' \in X$ such that $x \leq x'' \leq x'$, $k(x'') \geq k(x)$ and $k(x'') \geq k(x')$. Then $x'' \in D_i$ since D_i is a downset. Also, since $k(x') \in U_i$ by the choice of i, we get $k(x'') \in U_i$ because U_i is an upset, and similarly, we get $k(x'') \in U_j$ because $k(x)$ is in the upset U_j. Hence, $x'' \in D_i \cap k^{-1}(U_i \cap U_j)$, which is equal to $D_j \cap k^{-1}(U_i \cap U_j)$ by the assumption on the D_i's. Since D_j is a downset we get that $x \in D_j$, so $x \in D_j \cap k^{-1}(U_j)$, which implies that $x \in D$. □

As in Section 3.3 in Chapter 3, we associate to k an étale map $p : E_k \to (Y, \tau^\uparrow)$ and, hence, a sheaf F_k of distributive lattices over the space $(Y, \tau^\uparrow)$. By definition, the stalk of the sheaf F_k at $y \in Y$ is the quotient of A corresponding to the subspace $k^{-1}(\uparrow y)$.[10] Recall from Proposition 4.4.5 that $k^{-1}(\uparrow y) = C_y$, and by Proposition 4.4.1 the algebra corresponding to the subspace C_y is A / I_y. Hence, the stalk of F_k at a point y is the MV-algebraic quotient A / I_y. In particular, F_k is a sheaf of MV-algebras. Note that F_k is exactly the sheaf used by Yang in [142, Chapter 5] and by Dubuc and Poveda in [38, 2.1]. Thus, combining Proposition 3.3.6 in Chapter 3 with Proposition 4.6.1 above, we conclude the following.

---

[10]Indeed, note that, according to the definitions in Section 3.3, we have to use the specialization order of $(Y, \tau^\uparrow)$, which is the dual of the order $\leq$ on prime ideals.

Corollary 4.6.2 ([142, Proposition 5.1.2 and Remark 5.3.11] and [38, Theorem 3.12]). *Any MV-algebra is isomorphic to the MV-algebra of global sections of the sheaf over $(Y, \tau^\uparrow)$ whose stalk at $y \in Y$ is the quotient by the prime MV-ideal I_y.* □

By Remark 4.4.7 and Proposition 3.2.9.2, the algebra of local sections over a basic open set $\hat{a}^c \cap Y$ is the dual of $k^{-1}(\hat{a}^c)$; this is simply the quotient of A by the principal MV-ideal generated by a. In particular, the sheaf F_k is flasque on the basis B defined in Proposition 4.6.1.

We digress to compare our Proposition 4.6.1 to results in the literature. The following purely algebraic statement easily follows from the proof of Proposition 4.6.1 using duality.

Corollary 4.6.3 (Chinese Remainder Theorem for principal ideals). *Let $I_1, \ldots, I_n$ be finitely many principal MV-ideals of the MV-algebra A satisfying $\bigcap_{l=1}^{n} I_l = \{0\}$, and let $a_1, \ldots, a_n \in A$ be such that*

$$a_i \equiv a_j \mod I_i \oplus I_j$$

for all $i, j = 1, \ldots, n$. Then there exists a unique $b \in A$ such that

$$b \equiv a_l \mod I_l$$

for each $l = 1, \ldots, n$.

Sketch of Proof. Principal MV-ideals correspond to compact open subsets of $(Y, \tau^\uparrow)$, by Proposition 4.1.8 and the fact that $(Y, \tau^\uparrow) = (Y, \tau^\downarrow)^\partial$. Hence, $(I_l)_{l=1}^{n}$ yields a family $(U_l)_{l=1}^{n}$ of compact open subsets. Since $\bigcap_{l=1}^{n} I = \{0\}$, the family $(U_l)_{l=1}^{n}$ covers Y. The condition on $(a_l)_{l=1}^{n}$ says that their corresponding clopen downsets $(D_l)_{l=1}^{n}$ in the Priestley space $(X, \tau^p, \leq)$ satisfy the assumption of (P'_U) in the proof of Proposition 4.6.1. The proof of Proposition 4.6.1 now yields the desired b via its dual clopen downset D. □

In [45, Theorem 2.6] (cf. also [39, Lemma 1]), the authors prove a general Chinese Remainder Theorem for MV-algebras, where the restriction to principal ideals is not necessary. In the case of principal ideals, the proof exhibits a specific MV-algebraic expression for b in the elements $(a_l)_{l=1}^{n}$ and $(u_l)_{l=1}^{n}$, whenever the u_l's are elements of A that generate the principal MV-ideals $(I_l)_{l=1}^{n}$, i.e. satisfy $U_l = \hat{u}_l^c$ for each $l = 1, \ldots, n$. We will obtain such a stronger version in Corollary 4.6.5 below. We further observe that Dubuc and Poveda's Pullback-Pushout Lemma [38, 3.11], a key ingredient in their proof of the sheaf representation, is an easy consequence of the Chinese Remainder Theorem. In the work of Filipoiu and Georgescu on sheaf representations over the maximal MV-spectrum [46], the Chinese Remainder Theorem does not feature explicitly, though its implicit rôle is clear e.g. in

[46, Proposition 2.16]. In the literature on lattice-groups, already Keimel proved a Chinese Remainder Theorem [12, 10.6.3] in developing his sheaf representation; and Yang [142, Proposition 5.1.2] applies it to establish the sheaf representation via the co-compact dual of the spectral topology. It appears that Keimel's standard result, whose proof admits a straightforward translation in the language of MV-algebras, went unnoticed in much of the MV-algebraic literature. Indeed, [46], [38] and [39] do not refer to it. Finally, let us point out that sheaf representations and Chinese Remainder Theorems were studied at the level of universal algebra by Vaggione in [140], whose results extend the previous treatment by Krauss and Clark [95]; see also the earlier papers [141, 31] in the same direction.

In the proof of Proposition 4.6.1, we showed that the set D defined there was a clopen downset. Therefore, it must be of the form $\widehat{b}$ for a unique $b \in A$ by Priestley duality. In Corollary 4.6.5, we will exhibit an explicit MV-algebraic term for b. For this, we need one additional lemma concerning the map k.

Lemma 4.6.4. *For any $a, u \in A$, we have*

$$\widehat{a} \cap k^{-1}(\widehat{u}^c) \subseteq \bigcap_{m=0}^{\infty} \widehat{a \ominus mu} \subseteq \widehat{a} \cap \downarrow k^{-1}(\widehat{u}^c).$$

Proof. For the first inclusion, suppose that $x \in \widehat{a} \cap k^{-1}(\widehat{u}^c)$. We show by induction on m that $x \in \widehat{a \ominus mu}$ for all $m \geq 0$. For $m = 0$, we have $x \in \widehat{a}$ by assumption. Suppose that $x \in \widehat{a \ominus mu}$ for some m. We show that $x \in \widehat{a \ominus (m+1)u}$. Since $x \in k^{-1}(\widehat{u}^c)$, we have $u \in I_{k(x)}$. Therefore, by Proposition 4.4.6 and the assumption that $a \ominus mu \notin I_x$, we get $(a \ominus mu) \ominus u \notin I_x$. Now, since $(a \ominus mu) \ominus u \leq a \ominus (m+1)u$, we also have $a \ominus (m+1)u \notin I_x$, as required. For the second inclusion, suppose that $x \in \bigcap_{m=0}^{\infty} \widehat{a \ominus mu}$. Putting $m = 0$, it is clear that $x \in \widehat{a}$. To show that $x \in \downarrow k^{-1}(\widehat{u}^c)$, let

$$x' := x \overline{\oplus} \left(\bigvee_{m \geq 0} mu \right).$$

By Proposition 4.3.4.1, we have either $x' \in M^{\infty}$ or $x' = 1$. If $x' = 1$, then in particular $a \leq x'$. Compactness of the canonical extension yields $m \geq 0$ such that $a \leq x \overline{\oplus} mu$. By adjunction, $a \ominus mu \leq x$, which is impossible by assumption. We therefore conclude that $x' \in M^{\infty}$, and clearly $x \leq x'$. To show that $x' \in k^{-1}(\widehat{u}^c)$, note that

$$x' \overline{\oplus} u = x \overline{\oplus} \left(\bigvee_{m \geq 0} mu \right) \overline{\oplus} u = x \overline{\oplus} \left(\bigvee_{m \geq 0} (m+1)u \right) \leq x',$$

so $u \leq k(x')$, by definition of $k(x')$. $\qquad \square$

In the next corollary we write $[a]_y$ to denote the congruence class of $a \in A$ modulo I_y in the quotient MV-algebra A / I_y.

Corollary 4.6.5 (Term-definability). *Let $(u_i)_{i=1}^n$ and let $(a_i)_{i=1}^n$ be two finite collections of elements of A such that $(\widehat{u}_i^c)_{i=1}^n$ covers Y, and the a_i are compatible with this cover, i.e.,*

$$\text{For all } y \in Y, \text{ if } y \in \widehat{u}_i^c \cap \widehat{u}_j^c, \text{ then } [a_i]_y = [a_j]_y.$$

There exists an integer $t \geq 0$ such that, setting

$$b := \bigvee_{i=1}^n (a_i \ominus t u_i),$$

we have $[b]_y = [a_i]_y$ for all $y \in \widehat{u}_i^c \cap Y$.

Proof. By Proposition 4.6.1, $D := \bigcup_{i=1}^n (\widehat{a}_i \cap k^{-1}(\widehat{u}_i^c))$ is a clopen downset in Y. Fix $i \in \{1, \ldots, n\}$. Then $\widehat{a}_i \cap {\downarrow}k^{-1}(\widehat{u}_i^c) \subseteq D$, so $\bigcap_{m \in \mathbb{N}} \widehat{a_i \ominus m u_i} \subseteq D$ by Lemma 4.6.4. Note that, for $m \leq m'$, we have $a_i \ominus m' u_i \leq a_i \ominus m u_i$ using Lemma 4.1.4. Therefore, $(\widehat{a_i \ominus m u_i})_{m \in \mathbb{N}}$ is a decreasing chain of closed sets in τ^p. Since D is compact and it contains the full intersection of this chain, there exists $t_i \geq 0$ such that $\widehat{a_i \ominus t_i u_i} \subseteq D$. Choosing such t_i for each $i \in \{1, \ldots, n\}$ and setting $t := \max \{t_i \mid i = 1, \ldots, n\}$, we now have

$$D \subseteq \bigcup_{i=1}^n \left(\bigcap_{m \in \mathbb{N}} \widehat{a_i \ominus m u_i} \right) \subseteq \bigcup_{i=1}^n \left(\widehat{a_i \ominus t u_i} \right) \subseteq \bigcup_{i=1}^n \left(\widehat{a_i \ominus t_i u_i} \right) \subseteq D,$$

so $D = \bigcup_{i=1}^n \left(\widehat{a_i \ominus t u_i} \right)$. Now, putting $b = \bigvee_{i=1}^n (a_i \ominus t u_i)$, we get $\widehat{b} = D$, so that this b satisfies the required property, by Priestley duality and Corollary 4.6.2. $\square$

We now use the composite map $mk : X \to Z$ to obtain a sheaf representation over the maximal MV-spectrum. The space $(Z, \tau^{\downarrow})$ of maximal MV-ideals of A with the spectral topology is a compact Hausdorff space (cf. Proposition 4.5.4), and hence it is equal to its own co-compact dual (Example 2.1.9.3). The composite map $mk : X \to Z$ is continuous from $(X, \tau^{\downarrow})$ to $(Z, \tau^{\downarrow})$ by Lemma 4.5.3, so *a fortiori* it is continuous from (X, τ^p) to $(Z, \tau^{\downarrow})$. We also have the following.

Proposition 4.6.6. *The function $mk : X \to Z$ satisfies (P_U) in Definition 3.3.4 for $U := Z$.*

Proof. Let $(U_i)_{i \in I}$ be open in Z such that $Z = \bigcup_{i \in I} U_i$ and let $(D_i)_{i \in I}$ be clopen downsets in X such that $D_i \cap (mk)^{-1}(U_i \cap U_j) = D_j \cap (mk)^{-1}(U_i \cap U_j)$

for all $i, j \in I$. Since $(mk)^{-1}(Z) = X$, we must show that the set $D := \bigcup_{i \in I}(D_i \cap (mk)^{-1}(U_i))$ is a clopen downset. By continuity of mk, it is clear that D is an open downset in X. To see that D is moreover closed, notice that $D^c = \bigcup_{i \in I}((D_i)^c \cap (mk)^{-1}(U_i))$: the proof of this equality is, *mutatis mutandis*, the same as the proof of $(\star)$ in Proposition 4.6.1. Therefore, since the set $\bigcup_{i \in I}((D_i)^c \cap (mk)^{-1}(U_i))$ is clearly open, we conclude that D is closed. □

Remark 4.6.7. The function mk does not in general satisfy the condition (P_U) for all open sets of the form $U = \hat{a} \cap Z$. For example, consider the MV-algebra $C([0, 1], [0, 1])$ of continuous functions from $[0, 1]$ to $[0, 1]$. The maximal MV-spectrum of this algebra is homeomorphic to $[0, 1]$. If a is the identity function $[0, 1] \to [0, 1]$, then $U = \hat{a} \cap Z$ is the open set $(0, 1]$. The fact that (P_U) fails for this set U follows from the fact that there are continuous functions on $(0, 1]$ which can not be extended continuously to $[0, 1]$, for example, $x \mapsto \sin(\frac{1}{x})$. □

Again, we obtain an étale map $p \colon E_{mk} \to Z$ from the decomposition mk. The stalk of E_{mk} at $z \in Z$ is the quotient of A corresponding to the subspace $(mk)^{-1}(z)$, since the specialization order of Z is trivial. This is the MV-algebraic quotient given by the *germinal ideal* at z, as we will show now.

Proposition 4.6.8. *For any $z \in Z$, the closed subspace $(m \circ k)^{-1}(z)$ of X corresponds under Priestley duality to the MV-algebraic quotient $A/\mathfrak{o}_z$ of A, where the MV-ideal $\mathfrak{o}_z$ is defined by[11]*

$$\mathfrak{o}_z := \bigcap_{y \in \downarrow z \cap Y} I_y.$$

Proof. By Proposition 4.4.1, the quotient $A/\mathfrak{o}_z$ corresponds under Priestley duality to the subspace

$$S_{\mathfrak{o}_z} = \{x \in X \mid I_x \overline{\oplus} \mathfrak{o}_z \subseteq I_x\}.$$

Recall that, for any MV-ideal J, we have $I_x \oplus J \subseteq I_x$ if, and only if, $J \subseteq I_{k(x)}$, so we can write

$$S_{\mathfrak{o}_z} = \{x \in X \mid \mathfrak{o}_z \subseteq I_{k(x)}\}.$$

We will now prove that $S_{\mathfrak{o}_z} = (m \circ k)^{-1}(z)$. Let $x \in X$, and suppose first that $m \circ k(x) = z$. Then $I_{k(x)} \subseteq I_z$ by definition of m, so $\mathfrak{o}_z \subseteq I_{k(x)}$ by definition of $\mathfrak{o}_z$.

Conversely, suppose that $m \circ k(x) \neq z$. Then $k(x) \not\leq z$, since every prime MV-ideal is contained in a unique maximal ideal (Corollary 4.1.6), and also

---

[11]That is, the MV-ideal $\mathfrak{o}_z$ is the intersection of all prime MV-ideals contained in the maximal MV-ideal I_z. This is the *germinal MV-ideal at $z \in Z$*, see [121, Definition 4.7].

that $z \not\leq k(x)$, since z is a maximal MV-ideal. By Proposition 4.1.5, pick $u, v \in A$ such that $k(x) \in \hat{u}$, $z \in \hat{v}$ and $u \wedge v = 0$. If I_y is any prime MV-ideal which is contained in z, then in particular $0 \in I_y$ and $v \notin I_y$, so that $u \in I_y$. Hence, $u \in \mathfrak{o}_z$ by definition of $\mathfrak{o}_z$. However, by construction, we have $u \notin I_{k(x)}$, so $\mathfrak{o}_z \not\subseteq I_{k(x)}$, and therefore $x \notin S_{\mathfrak{o}_z}$. □

By the preceding proposition, the stalk of the étalé space E_{mk} at $z \in Z$ is the MV-algebra $A / \mathfrak{o}_z$. This is exactly the étalé space used by Filipoiu and Georgescu in [46]. Applying Lemmas 3.3.5 and 3.3.3 in light of Proposition 4.6.6, we obtain the following result.

Corollary 4.6.9 ([46, Proposition 2.16]). *Any MV-algebra is isomorphic to the MV-algebra of global sections of the sheaf over $(Z, \tau^{\downarrow})$ whose stalk at $z \in Z$ is the quotient by the germinal MV-ideal $\mathfrak{o}_z$.* □

Remark 4.6.10. As shown in Section 4.5, the sets $(mk)^{-1}(z)$, for $z \in Z$, are the order-components of $(X, \leq)$, and are in fact also the equivalence classes of the equivalence relation W defined there. Moreover, every W-equivalence class contains exactly one point of Z, and the space X/W is homeomorphic to Z, as was proved in Proposition 4.5.4. □

Concluding remarks

In this chapter, we have applied the methods of Chapter 3 to MV-algebras. The same methods could in principle be applied to any variety of algebras with a distributive lattice order. It would be interesting to obtain sheaf-theoretic representations for other significant classes of distributive-lattice-ordered algebras in this way.

Our Proposition 4.3.7 determines enough of the structure of the dual Stone-Priestley space of an MV-algebra to establish our main results on sheaf representations. It has been a long-standing open problem in the theory of lattice-ordered abelian groups to characterise spectra of prime congruences in topological terms, see [27, and references therein]. The problem of characterizing the dual Stone-Priestley space of a unital lattice-ordered abelian group is likewise open, to the best of our knowledge. In light of Proposition 4.3.7, we wonder whether a characterisation that uses the partial semigroup structure described there is possible. A further question of interest is to investigate the functoriality of the construction under appropriate duals of MV-algebraic homomorphisms.

Theorem 4.5.5, the MV-algebraic generalization of Kaplansky's theorem, shows that one can construct the maximal MV-spectrum of any MV-algebra from its underlying lattice only. Example 4.5.7 shows that this fails for prime MV-spectra. The question arises whether there are significant classes

of MV-algebras with the property that the prime MV-spectrum is uniquely determined by the underlying lattice of the MV-algebra. Recall that an algebra in a variety is called *simple* if it has no non-trivial congruences, and *semisimple* if it is a subdirect product of simple algebras. Semisimple MV-algebras are precisely those that arise as subalgebras of the MV-algebra $C(S, [0, 1])$ of all continuous $[0, 1]$-valued functions on a compact Hausdorff space S, see [25, 3.6.8]. Example 4.5.7 uses a non-semisimple MV-algebra. An MV-algebra that is semisimple, and is such that each of its quotient algebras also has that property, is known as *hyperarchimedean* in the MV-algebraic literature [25, 6.3]. Equivalently, an MV-algebra is hyperarchimedean just in case it has no non-maximal prime MV-ideals [25, 6.3.2]. Therefore the underlying lattice of a hyperarchimedean MV-algebra determines its prime MV-spectrum to within a homeomorphism by Theorem 4.5.5. We wonder whether the semisimple property, or a strengthening thereof that is weaker than the hyperarchimedean property, suffices to guarantee that the prime MV-spectrum of an MV-algebra be uniquely determined by its underlying lattice.

Chapter 5. A non-commutative Priestley duality

In this chapter, we prove that the category of left-handed strongly distributive skew lattices is dually equivalent to a category of sheaves of sets over local Priestley spaces. Our result thus provides a non-commutative version of classical Priestley duality for distributive lattices. This result also generalizes the recent development of Stone duality for skew Boolean algebras. From the point of view of skew lattices, Leech showed early on that any strongly distributive skew lattice can be embedded in the skew lattice of partial functions on some set with the operations being given by restriction and so-called override. Our duality shows that there is a canonical choice for this embedding. Conversely, from the point of view of sheaves over Boolean spaces, our results show that skew lattices correspond to Priestley orders on these spaces and that skew lattice structures are naturally appropriate in any setting involving sheaves over Priestley spaces. This chapter is a modified version of the paper [8].

Skew lattices [104, 105] are a non-commutative version of lattices: algebraically, a skew lattice is a structure $(S, \vee, \wedge)$, where $\vee$ and $\wedge$ are binary operations which satisfy the associative and idempotent laws, and certain absorption laws. A (proto)typical example of a skew lattice is the collection of partial functions from a set X to a set Y, equipped with two binary operations called *restriction* and *override*. Here, the restriction of f by g is defined as the function which takes the value of f only if both f and g are defined, and the override of f by g is the function which takes the value of g if it is defined, and the value of f otherwise. Skew lattices of partial functions always have a *zero* element, namely the function with empty domain. In addition, we will see that they satisfy two additional axioms, namely *strong distributivity*, which generalizes the usual distributive law for lattices, and *left-handedness*, cf. Section 5.1 below. It is a consequence of the results in this chapter that every left-handed strongly distributive skew lattice with zero can be embedded into a skew lattice of partial functions. This fact was first proved in [106, 3.7] as a consequence of the description of the subdirectly irreducible algebras in the variety of strongly distributive skew lattices. Our proof will not depend on this result, and it will moreover provide a canonical choice of an enveloping skew lattice of partial functions. A related result in computer science is described in [10], where the authors give a complete axiomatisation of the structure of partial functions with the

operations override and 'update', from which the 'restriction' given above can also be defined.

In this chapter, we generalize the above idea of 'skew lattices of partial functions' to 'skew lattices of local sections of a sheaf'. If F is a sheaf over a Boolean space X, then the set of all local sections with clopen domains forms a skew Boolean algebra [7, 96]. If X is moreover equipped with a partial order that makes it into a Priestley space, then the local sections with domains that are clopen downsets form a left-handed strongly distributive skew lattice. It follows from our duality in this chapter that this accounts, up to isomorphism, for *all* such skew lattices: we will prove that *every left-handed strongly distributive skew lattice with zero is isomorphic to a skew lattice of all local sections over clopen downsets of some bundle*. Thus, in any setting where sheaves over Priestley spaces are present, strongly distributive skew lattice structures are intrinsic, in addition to whatever other structure the stalks of the sheaf may be equipped with. Moreover, it will be a consequence of our duality result that there is a canonical choice for the bundle and base space that represent a given skew lattice. We will prove that, among all representing bundles for a given skew lattice, there is a unique bundle $p : E \to X$ such that p is a local homeomorphism and X is a local Priestley space (i.e., a space whose one-point-compactification is a Priestley space). This result generalizes both Priestley duality, and recent results on Stone duality [137] for skew Boolean algebras [7, 96, 98, 97].

In conclusion, our results show that the embeddability of strongly distributive skew lattices in partial function algebras is not coincidental, but a fully structural and natural phenomenon. They also show that strongly distributive skew lattices are intrinsic to sheaves over Priestley spaces and that each such lattice has a canonical embedding into a skew Boolean algebra, namely the skew Boolean algebra of all local sections with clopen domains over the corresponding base. Thus our results open the way to exploring the logic of such structures. In particular, they provide a candidate notion of Booleanization, which may in turn lead to the development of a non-commutative version of Heyting algebras; also see the concluding remarks in this chapter.

Outline of the chapter. We first provide background on skew lattices (Section 5.1), and recall a 'local' version of Priestley duality, which holds for distributive lattices that do not necessarily have a largest element (Section 5.2). After these preliminaries, we will be ready to state our main theorem (Theorem 5.2.6), that the categories of left-handed strongly distributive skew lattices and sheaves over local Priestley spaces are dually equivalent. Starting the proof of this theorem, we first describe the functor which associates a skew lattice of local sections to a sheaf (Section 5.3). To show that this func-

tor is part of a dual equivalence, we will describe how to reconstruct the sheaf from its skew lattice of local sections (Section 5.4), and give a general description of this process for an arbitrary left-handed strongly distributive skew lattice. Finally (Section 5.5), we will put together the results from the preceding sections to prove our main theorem.

5.1. Strongly distributive left-handed skew lattices

For an extensive introduction to the theory of skew lattices we refer the reader to [104, 105, 106, 107]. To make our exposition self-contained, in this section we collect some definitions and basic facts of the theory.

Definition 5.1.1. A *skew lattice* S is an algebra $(S, \wedge, \vee, 0)$ of type $(2, 2, 0)$, such that the operations $\wedge$ and $\vee$ are associative, idempotent and satisfy the absorption identities

$$x \wedge (x \vee y) = x = x \vee (x \wedge y),$$
$$(y \vee x) \wedge x = x = (y \wedge x) \vee x,$$

and the 0 element satisfies $x \wedge 0 = 0 = 0 \wedge x$.

Notice that it follows from the absorption laws that 0 is a neutral element for $\vee$. Also note that a lattice is a skew lattice in which $\wedge$ and $\vee$ are commutative.

Remark 5.1.2. In this chapter, the terms "skew lattice" and "lattice" will always be understood to mean "skew lattice with zero" and "lattice with zero". However, throughout this chapter, in contrast with all other chapters in this thesis, neither skew lattices nor lattices are assumed to have an element 1 that is absorbent for $\vee$.

The *partial order* $\leq$ on a skew lattice S is defined by

$$x \leq y \iff x \wedge y = x = y \wedge x,$$

which is equivalent to $x \vee y = y = y \vee x$, by the absorption laws. Note that 0 is the minimum element in the partial order $\leq$. If S and T are skew lattices, we say a function $h : S \to T$ is a *homomorphism* if it preserves the operations $\wedge$, $\vee$ and the zero element.

The full inclusion of lattices into skew lattices has a left adjoint, which can be explicitly defined using Green's equivalence relation $\mathcal{D}$, which is well known in semigroup theory [73, 81]. Recall that $\mathcal{D}$ is the equivalence relation on a skew lattice S defined by $x \mathcal{D} y$ if, and only if, $x \wedge y \wedge x = x$ and $y \wedge x \wedge y = y$, or equivalently, $x \vee y \vee x = x$ and $y \vee x \vee y = y$. The following is a version of Leech's "first decomposition theorem for skew lattices" [104].

Theorem 5.1.3 ([104], 1.7). *Let S be a skew lattice. The relation $\mathcal{D}$ is a congruence, and $\alpha_S : S \to S/\mathcal{D}$ is a lattice quotient of S. For any homomorphism $h : S \to L$ where L is a lattice, there exists a unique homomorphism $\bar{h} : S/\mathcal{D} \to L$ such that $\bar{h} \circ \alpha_S = h$.*

In particular, any skew lattice homomorphism $h : S \to T$ induces a homomorphism between the lattice reflections of S and T, which is defined as the unique factorization of the composite map $\alpha_T \circ h : S \to T/\mathcal{D}$. By a slight abuse of notation, we will also denote this homomorphism by $\bar{h}$, which is then a map from $S/\mathcal{D}$ to $T/\mathcal{D}$. In the context of lattices which may not have a largest element, the notion of *proper* homomorphism [37] is pertinent.

Definition 5.1.4. A lattice homomorphism $f : L_1 \to L_2$ is called *proper* provided that for any $y \in L_2$ there is some $x \in L_1$ such that $f(x) \geq y$. We call a skew lattice homomorphism $h : S \to T$ *proper* if its lattice reflection $\bar{h}$ is proper.

Note that a lattice homomorphism between *bounded* lattices is proper if, and only if, it preserves the largest element.

There are several non-equivalent generalizations of the notion of 'distributivity' for lattices to the non-commutative setting. The objects of study in this chapter are *left-handed strongly distributive skew lattices*[1].

Definition 5.1.5. A skew lattice is called *strongly distributive* if it satisfies the identities

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \tag{5.1}$$
$$(y \vee z) \wedge x = (y \wedge x) \vee (z \wedge x). \tag{5.2}$$

A skew lattice is called *left-handed* if it satisfies the identity

$$x \wedge y \wedge x = x \wedge y, \text{ or, equivalently, } x \vee y \vee x = y \vee x.$$

The notion of *right-handed* skew lattices is defined dually.

Note that, if S is a strongly distributive skew lattice, then $S/\mathcal{D}$ is a distributive lattice. We recall Leech's second decomposition theorem, which relates left-handed and right-handed skew lattices to general skew lattices. Recall that Green's equivalence relation $\mathcal{R}$ for the operation $\wedge$ of a skew lattice S is defined by $x\mathcal{R}y$ if, and only if, $x \wedge y = y$ and $y \wedge x = x$. Dually, $\mathcal{L}$ is defined on S by $x\mathcal{L}y$ if, and only if, $x \wedge y = x$ and $y \wedge x = y$.

---

[1]Note that what we call a strongly distributive skew lattice here is called a *meet bidistributive and symmetric skew lattice* in the terminology of [106].

Theorem 5.1.6 ([104], Theorem 1.15). *The relations $\mathcal{L}$ and $\mathcal{R}$ are congruences for any skew lattice S. Moreover, $S/\mathcal{L}$ is the maximal right-handed image of S, $S/\mathcal{R}$ is the maximal left-handed image of S, and the following diagram is a pull-back:*

$$
\begin{array}{ccc}
S & \longrightarrow & S/\mathcal{R} \\
\downarrow & & \downarrow \\
S/\mathcal{L} & \longrightarrow & S/\mathcal{D}
\end{array}
$$

Left-handed strongly distributive skew lattices have some desirable algebraic properties that we collect here, for use in what follows.

Lemma 5.1.7. *Let S be a left-handed strongly distributive skew lattice, and let $a, a', b \in S$.*

1. *The semigroup $(S, \wedge)$ is left normal, i.e., $b \wedge a \wedge a' = b \wedge a' \wedge a$.*

2. *If $a, a' \leq b$ and $[a]_{\mathcal{D}} = [a']_{\mathcal{D}}$, then $a = a'$.*

Proof. For the first item, recall that strongly distributive skew lattices are *normal*, i.e., they satisfy the equation $b \wedge a \wedge a' \wedge b = b \wedge a' \wedge a \wedge b$, see e.g. [106, Theorem 2.5]. Left normality then easily follows from left-handedness. For the second item, note that $a \mathcal{D} a'$ and left-handedness together yield $a \wedge a' = a$ and $a' \wedge a = a'$. Therefore, since $a \leq b$, we get $a = b \wedge a = b \wedge a \wedge a'$, and similarly $a' = b \wedge a' \wedge a$. Using the first item, we conclude that $a = a'$. $\qquad\square$

In what follows, primitive skew lattices will play an important role. A skew lattice S is called *primitive* if it has only one non-zero $\mathcal{D}$-class, or, equivalently, if $S/\mathcal{D}$ is the two-element distributive lattice, **2**. For any set T, there is a unique primitive left-handed skew lattice P_T with T as its only non-zero $\mathcal{D}$-class. The operations inside this $\mathcal{D}$-class are determined by lefthandedness: $t \wedge t' = t$ and $t \vee t' = t'$, for any $t, t' \in T$. Note that P_T is strongly distributive.

P_T
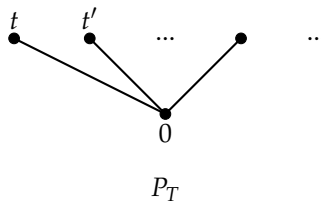
5.2. Sheaves over local Priestley spaces

In this section we first recall a slight modification of Priestley duality, which is available for distributive lattices that may not have a largest element. We then define the category of sheaves over local Priestley spaces, and state our main theorem.

We denote by $\mathbf{DL}_0^h$ the category of distributive lattices with a zero element and lattice homomorphisms, and by $\mathbf{DL}_0$ the (non-full) subcategory containing only the proper lattice homomorphisms (cf. Definition 5.1.4). Note that the category $\mathbf{DL}_0^h$ is dually equivalent to the category $\mathbf{PS}^\infty$ of Priestley spaces with a largest element and continuous monotone maps between them. Here, the largest element, ∞, represents the constantly zero homomorphism into $\mathbf{2}$, the empty prime filter, or the full prime ideal. To obtain a duality for the subcategory $\mathbf{DL}_0$ of $\mathbf{DL}_0^h$, we now reason as follows. For objects D, E of $\mathbf{DL}_0$, the proper homomorphisms correspond to those morphisms in $\mathbf{PS}^\infty$ for which $f^{-1}(\infty) = \{\infty\}$. Therefore, if we only want to represent *proper* homomorphisms, we can safely remove the largest element ∞ from the Priestley dual space X of a lattice D, to obtain a space X'. The original Priestley space X can be recovered as the ordered one-point compactification of X'. This leads to the following definition.

Definition 5.2.1. We say that $(X, \tau, \leq)$ is a *local Priestley space* if its ordered one-point-compactification $(X^\infty, \tau^\infty, \leq)$, with ∞ a new largest element, is a Priestley space. In the *category* **LPS** *of local Priestley spaces*, a morphism $f : (X, \tau_X, \leq_X) \to (Y, \tau_Y, \leq_Y)$ is the restriction of a continuous monotone map between the one-point-compactifications $f : \widehat{X} \to \widehat{Y}$ for which $f^{-1}(\infty_Y) = \{\infty_X\}$.

Remark 5.2.2. It is possible to give an equivalent definition of the category **LPS** without referring to the ordered one-point-compactification: local Priestley spaces are exactly the totally order-disconnected spaces for which the space $(X, \tau^\downarrow)$ has a basis consisting of τ-compact open downsets, and **LPS**-morphisms $(X, \tau_X, \leq_X) \to (Y, \tau_Y, \leq_Y)$ are equivalently described as continuous monotone maps with the further property that the inverse image of a τ_Y-compact set is τ_X-compact.

There is a dual equivalence between **LPS** and $\mathbf{DL}_0$, that can be described as follows. Let $D_* := \mathbf{DL}_0^h(D, \mathbf{2}) \setminus \{\infty\}$, which is equal to $\mathbf{DL}_0(D, \mathbf{2})$, since the only non-proper homomorphism $D \to \mathbf{2}$ is ∞. Then D_* is a local Priestley space, and the duals of *proper* homomorphisms $D \to E$ restrict correctly to functions $E_* \to D_*$, by the arguments given above. Conversely, if $(X, \tau, \leq)$ is a local Priestley space, let $(X, \tau, \leq)^*$ be the distributive lattice of clopen proper downsets of the ordered one-point-compactification of $(X, \tau, \leq)$, or

equivalently, compact open downsets of $(X, \tau, \leq)$; $(-)^*$ extends to a functor in the obvious manner. We then have the following corollary to Priestley duality.

Corollary 5.2.3. *The contravariant functors* $(-)_* : \mathbf{DL}_0 \rightarrow \mathbf{LPS}$ *and* $(-)^* : \mathbf{LPS} \rightarrow \mathbf{DL}_0$ *establish a dual equivalence between the categories* $\mathbf{DL}_0$ *and* $\mathbf{LPS}$.

We now define the category of sheaves over local Priestley spaces and state our main result. We refer to Section 3.1 in Chapter 3 for the definitions of sheaf and étalé space, and the correspondence between them.

Remark 5.2.4. In this chapter, contrary to earlier chapters, we will only be concerned with sheaves and étalé spaces *of sets*; sheaves and étalé spaces of distributive lattices will not play a role. We will interchangably use the equivalent descriptions of a sheaf as a functor and as an étale map, cf. Theorem 3.1.3. Throughout this chapter, "étale map" will be understood to mean "surjective local homeomorphism", and "sheaf" will be understood to mean "sheaf whose associated étale map is surjective". Note that such sheaves do not necessarily have global sections. This agrees with the fact that the skew lattices in this chapter do not necessarily have a largest $\mathcal{D}$-class.

If E is a sheaf on a topological space X and $f : X \rightarrow Y$ is a continuous map, recall that the direct image sheaf f_*E over Y is defined by letting the set of sections over $V \subseteq Y$ be $E(f^{-1}(V))$. We will denote by $\mathbf{Sh}(\mathbf{LPS})$ the category of *sheaves over local Priestley spaces*: an *object* is $(X, \tau, \leq, E)$, where $(X, \tau, \leq)$ is a local Priestley space, and E is a sheaf. A *morphism* from $(X, \tau, \leq, E)$ to $(Y, \tau, \leq, F)$ is a pair (f, λ), where f is an $\mathbf{LPS}$-morphism $(X, \tau, \leq) \rightarrow (Y, \tau, \leq)$, and $\lambda : F \Rightarrow f_*E$ is a natural transformation; see the diagram in Figure 5.1. If $(f, \lambda) : (X, E) \rightarrow (Y, F)$ and $(g, \mu) : (Y, F) \rightarrow (Z, G)$ are morphisms in $\mathbf{Sh}(\mathbf{LPS})$, their composition is defined by (gf, ν), where $\nu_U := \lambda_{g^{-1}(U)} \circ \mu_U$.

$$E \qquad f_*E \xleftarrow{\quad \lambda \quad} F$$
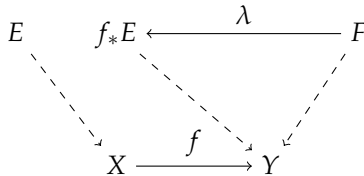
Figure 5.1: A morphism in the category $\mathbf{Sh}(\mathbf{LPS})$.

In the proof of Proposition 5.5.1, we will use the following lemma.

Lemma 5.2.5. *Suppose (f, λ) and (f, λ') are morphisms from a sheaf E on X to a sheaf F on Y, and suppose that $\mathcal{B}$ is a basis for the space Y. If, for all $V \in \mathcal{B}$, $\lambda_V = \lambda'_V$, then $\lambda = \lambda'$.*

We are now ready to state our main theorem; the rest of this chapter will be devoted to its proof.

Theorem 5.2.6. *The category* **SDL** *of left-handed strongly distributive skew lattices is dually equivalent to the category* **Sh(LPS)** *of sheaves over local Priestley spaces.*

5.3. The functor from spaces to algebras

Let $X = (X, \tau, \leq)$ be a local Priestley space and let $p : E \to X$ be a bundle over X, i.e., a continuous function to X. The sheaf of local sections of p is a functor $\Omega(X)^{\mathrm{op}} \to \mathbf{Set}$, which we also denote by the letter E. Let $L := X^*$ be the distributive lattice of compact open downsets of X. We define a skew lattice structure on the set $S := \bigsqcup_{U \in L} E(U)$, that is, the set of all local sections over all compact open downsets of X. Let $U, V \in L$ and $a \in E(U)$, $b \in E(V)$. We define the *override* $a \vee b$ to be the local section over $U \cup V$ given by

$$(a \vee b)(x) := \begin{cases} b(x), & \text{if } x \in V, \\ a(x), & \text{if } x \in U \setminus V. \end{cases} \tag{5.3}$$

Note that this indeed defines a continuous section from $U \cup V$ to E, so that $a \vee b \in E(U \cup V)$. The section $a \vee b$ is the patch of the compatible family consisting of the two elements $a|_{U \setminus V}$ and $b|_V$. We define the *restriction* $a \wedge b$ to be the local section over $U \cap V$ defined by

$$(a \wedge b)(x) := a(x) \text{ for all } x \in U \cap V, \tag{5.4}$$

i.e., $a \wedge b := a|_{U \cap V}$. Note that the unique section over the empty set is a zero element for this operation $\wedge$, and we shall therefore denote it by 0. In the following proposition, we collect some basic properties of the algebra S that we constructed here.

Proposition 5.3.1. *Let $p : E \to X$ be a bundle over a local Priestley space, $L := X^*$. Let $S = \bigsqcup_{U \in L} E(U)$ be the algebra defined in (5.3) and (5.4). Then the following hold.*

1. *The algebra S is a left-handed strongly distributive skew lattice.*

2. *The lattice reflection $S/\mathcal{D}$ of S is isomorphic to L.*

3. *The partial order on S is given by $a \leq b$ if, and only if, the section a is a restriction of b.*

Proof. It is known [106] and easy to check that the set $\mathcal{P}(X, E)$ of all partial maps from X to E, equipped with operations defined as in (5.3) and (5.4), is a left-handed strongly distributive skew lattice. Since S is a subalgebra of $\mathcal{P}(X, E)$, it is also a left-handed strongly distributive skew lattice, which proves item (1). For item (2), note that the relation $\mathcal{D}$ on S is given by $a\mathcal{D}b$ if, and only if, $\mathrm{dom}(a) = \mathrm{dom}(b)$. Hence, $S/\mathcal{D}$ is isomorphic to the lattice of domains, L. Item (3) follows from the definitions of the partial order and the operation $\wedge$. $\qquad\square$

So far, we only needed to assume that $p : E \to X$ was a bundle. To extend the above construction to a contravariant functor from $\mathbf{Sh}(\mathbf{LPS})$ to $\mathbf{SDL}$, it will be convenient to work with étale maps. For any étale map $p : E \to X$, we denote by $(E^\star, \vee, \wedge, 0)$ the *dual algebra* of local sections defined above. Suppose that $p : E \to X$ and $q : F \to Y$ are étale maps over local Priestley spaces, and that (f, λ) is a morphism from the sheaf E to the sheaf F, as in Figure 5.1 above. We define a skew lattice morphism $(f, \lambda)^\star : F^\star \to E^\star$. Let $a \in F^\star$, so $a \in F(U)$ for some compact open downset U of Y. Then $f^{-1}(U)$ is a compact open downset of X. We now define $(f, \lambda)^\star(a) := \lambda_U(a)$, which is an element of $f_* E(U) = E(f^{-1}(U))$, i.e., a section over $f^{-1}(U)$.

Lemma 5.3.2. *The function* $(f, \lambda)^\star : F^\star \to E^\star$ *is a morphism in* $\mathbf{SDL}$, *for which the lattice reflection* $\overline{(f, \lambda)^\star}$ *is equal to* f^*.

Proof. Let us write h for the function $(f, \lambda)^\star$. We show in detail that h preserves the operation $\wedge$, and leave it to the reader to verify that h preserves $\vee$ and 0, since the proofs are similar. Let $a \in F(U)$, $b \in F(V)$. By definition of $\wedge$, $h(a) \wedge h(b)$ is $h(a)|_{f^{-1}(U) \cap f^{-1}(V)}$. By naturality of λ, the following diagram commutes:

$$
\begin{array}{ccc}
F(U) & \xrightarrow{\ \ \lambda_U\ \ } & f_* E(U) \\[2pt]
{\scriptstyle (-)|_{U \cap V}}\Big\downarrow & & \Big\downarrow{\scriptstyle (-)|_{f^{-1}(U \cap V)}} \\[2pt]
F(U \cap V) & \xrightarrow[\ \lambda_{U \cap V}\ ]{} & f_* E(U \cap V)
\end{array}
$$

In particular, we get

$$
h(a \wedge b) = \lambda_{U \cap V}(a \wedge b) = \lambda_{U \cap V}(a|_{U \cap V}) = \lambda_U(a)|_{f^{-1}(U \cap V)}
$$
$$
= h(a)|_{f^{-1}(U) \cap f^{-1}(V)} = h(a) \wedge h(b).
$$

Further note that $\overline{h} : F^\star/\mathcal{D} \to E^\star/\mathcal{D}$ is exactly the proper homomorphism that is dual to f in Priestley duality. Therefore, h is a morphism in $\mathbf{SDL}$ and $\overline{h} = f^*$. $\qquad\square$

In conclusion, we record the following proposition.

Proposition 5.3.3. *The assignments $(E, p, X) \mapsto (E, p, X)^\star$ and $(f, \tau) \mapsto (f, \tau)^\star$ define a contravariant functor $(-)^\star$ from $\mathbf{Sh}(\mathbf{LPS})$ to $\mathbf{SDL}$.*

Proof. By Proposition 5.3.1.1 and Lemma 5.3.2, the assignments are well-defined. We leave functoriality to the reader. ☐

5.4. Reconstructing a space from its dual algebra

In this section, we show how a sheaf E over a local Priestley space X can be reconstructed (up to homeomorphism) from its dual algebra $E^\star$, defined in the previous section.

Fix a surjective étale map $p : E \twoheadrightarrow X$ over a local Priestley space X. Note that the assumptions that p is surjective and that X is a local Priestley space imply that, for any $x \in X$, there exists a section of p over a compact open downset that contains x. Let $E^\star$ be the left-handed strongly distributive skew lattice associated to E in Proposition 5.3.3, and let $\alpha : E^\star \to L$ be the lattice reflection of $E^\star$ (cf. Theorem 5.1.3). Note that, by Proposition 5.3.1.1, the lattice reflection L of $E^\star$ is isomorphic to X^*, so that X is homeomorphic to the Priestley dual space L_* of L. By Theorem 5.1.3, the set $\mathbf{DL}_0(L, \mathbf{2})$ that underlies the space L_* is in a natural bijection with $\mathbf{SDL}(E^\star, \mathbf{2})$. Note that the composite bijection $X \to \mathbf{SDL}(E^\star, \mathbf{2})$ sends $x \in X$ to the homomorphism $h_x : E^\star \to \mathbf{2}$ defined by

$$h_x(a) := \begin{cases} 1 & \text{if } x \in \text{dom}(a) \\ 0 & \text{otherwise.} \end{cases} \tag{5.5}$$

Obviously, the bijection $x \mapsto h_x$ can be extended to an order-homeomorphism by translating the order and topology of X to $\mathbf{SDL}(E^\star, \mathbf{2})$. We thus naturally equip the set $\mathbf{SDL}(E^\star, \mathbf{2})$ with the reverse pointwise order, and the topology generated by sets of the form $\{h \in \mathbf{SDL}(E^\star, \mathbf{2}) : h(a) = 1\}$ and their complements, where a ranges over $E^\star$.

We will now show that the algebraic structure of $E^\star$ is enough to reconstruct, for any $x \in X$, the stalk E_x of the sheaf E at x. To this end, fix $x \in X$ and write P_x for the primitive skew lattice whose non-zero $\mathcal{D}$-class is the set E_x. We define a natural evaluation homomorphism $\text{ev}_x : E^\star \to P_x$ by

$$\text{ev}_x(a) := \begin{cases} a(x) & \text{if } x \in \text{dom}(a) \\ 0 & \text{otherwise.} \end{cases}$$

Note that the composition $\alpha \circ \text{ev}_x : E^\star \to \mathbf{2}$ is exactly the map h_x naturally associated to x in (5.5). We now characterize the kernel of the homomorphism ev_x by an algebraic property which only refers to the map h_x and the skew lattice operations of $E^\star$, as follows.

Lemma 5.4.1. *Let $x \in X$. For any $a, b \in E^\star$, the following are equivalent:*

1. $\mathrm{ev}_x(a) = \mathrm{ev}_x(b)$;

2. *there exist $c, d \in E^\star$ such that $h_x(c) = 0$, $h_x(d) = 1$, and*
   $(a \wedge d) \vee c = (b \wedge d) \vee c$.

Proof. It is obvious from the definitions of $\wedge$ and $\vee$ that (2) implies (1). To see that (1) implies (2), suppose that $\mathrm{ev}_x(a) = \mathrm{ev}_x(b)$. Note that, in particular, $h_x(a) = h_x(b)$, so that $x \in \mathrm{dom}(a)$ if, and only if, $x \in \mathrm{dom}(b)$. Therefore, either (i) $x \in \mathrm{dom}(a) \cap \mathrm{dom}(b)$, or (ii) $x \notin \mathrm{dom}(a) \cup \mathrm{dom}(b)$. In case (i), we have $a(x) = \mathrm{ev}_x(a) = \mathrm{ev}_x(b) = b(x)$, so that x is in the equalizer $\|a = b\|$, i.e., the set of elements in $\mathrm{dom}(a) \cap \mathrm{dom}(b)$ for which a and b take the same value. Note that $\|a = b\|$ is open in X because a and b are continuous. Since X is a local Priestley space, there exist compact open downsets U and V of X such that $x \in U \setminus V \subseteq \|a = b\|$. Using that the étale map is surjective, pick $d \in E^\star$ with $x \in \mathrm{dom}(d) \subseteq U$. Again by surjectivity of the étale map, note that the set V can be covered by compact open downsets which are domains of local sections. Therefore, by compactness of V, there exist $c_1, \ldots, c_n \in E^\star$ whose domains cover V. Let c be the section defined by $\bigvee_{i=1}^n c_i$, so that $\mathrm{dom}(c) = V$. By construction, we have $h_x(c) = 0$ and $h_x(d) = 1$. It is easy to verify, using the definitions of $\wedge$ and $\vee$, that $(a \wedge d) \vee c = (b \wedge d) \vee c$. In case (ii), pick any section d with $x \in \mathrm{dom}(d)$ and define $c := a \vee b$. It is again easy to check that c and d satisfy the requirements of (2). $\qquad\square$

Hence, given a point $x \in X$, we define a relation $\sim_x$ on $E^\star$ by

$$a \sim_x b \iff \exists c, d \in S : h_x(c) = 0, h_x(d) = 1, \text{ and } (a \wedge d) \vee c = (b \wedge d) \vee c,$$

and we immediately obtain:

Proposition 5.4.2. *Let $x \in X$. The relation $\sim_x$ is a skew lattice congruence on $E^\star$, and there is an isomorphism between $E^\star / \sim_x$ and P_x that takes the quotient map $E^\star \twoheadrightarrow E^\star / \sim_x$ to the evaluation map $\mathrm{ev}_x : E^\star \twoheadrightarrow P_x$.*

Proof. The preceding lemma exactly shows that $\sim_x$ is the kernel of the morphism ev_x. The result now follows from the first isomorphism theorem of universal algebra. $\qquad\square$

For a primitive skew lattice P, we denote by P^1 the unique non-zero $\mathcal{D}$-class of P, considered as a set.

Corollary 5.4.3. *The étalé space $p : E \twoheadrightarrow X$ is isomorphic to $q : (E^\star)_\star \twoheadrightarrow X$, where*

- *the set underlying the space $(E^\star)_\star$ is*

$$\bigsqcup_{x \in X} (E^\star/\sim_x)^1 = \{(x, [a]_{\sim_x}) : x \in X, [a]_{\sim_x} \in (E^\star/\sim_x)^1\},$$

- *the function $q : (E^\star)_\star \twoheadrightarrow X$ sends an element of the disjoint union to its index $x \in X$,*

- *the topology on $(E^\star)_\star$ is given by taking as a basis of open sets the sets of the form*

$$\widehat{a} := \{(x, [a]_{\sim_x}) \mid x \in \mathrm{dom}(a)\},$$

where a ranges over the elements of $E^\star$.

Proof. Define a map $\psi : E \to (E^\star)_\star$ by sending $e \in E_x$ to $(x, [a]_{\sim_x})$, where a is any local section for which $a(x) = e$; such a section exists because p is an étale map, and the value of $\psi(e)$ does not depend on the choice of a because of Lemma 5.4.1. By Proposition 5.4.2, ψ is a bijection. It is not hard to see from the definition of the topologies on E and $(E^\star)_\star$ that ψ is open and continuous. Hence, ψ is a homeomorphism, and ψ clearly commutes with the étale maps. $\qquad\square$

We now generalize the above construction to an *arbitrary* left-handed strongly distributive skew lattice S. This is the main contribution of this chapter, and it is the key to the proof that the functor $(-)^\star$ defined in Section 5.3 is part of a contravariant equivalence of categories.

Let S be a left-handed strongly distributive skew lattice. Inspired by the above results, for any proper homomorphism $h : S \to \mathbf{2}$, we define a relation $\sim_h$ as follows:

$$a \sim_h b \iff \exists c, d \in S : h(c) = 0, h(d) = 1, \text{ and } (a \wedge d) \vee c = (b \wedge d) \vee c.$$

The following proposition is the central technical result that we need to construct the étalé space dual to the skew lattice S.

Proposition 5.4.4. *Let S be a left-handed strongly distributive skew lattice, and $h : S \to \mathbf{2}$ a proper homomorphism.*

1. *The relation $\sim_h$ is a skew lattice congruence on S which refines $\ker(h)$.*

2. *The quotient skew lattice $S/\sim_h$ is primitive and the diagram*

commutes.

3. *For any commuting diagram in* **SDL** *of the form*

$$
\begin{array}{ccc}
S & & \\
\sigma \downarrow & \searrow h & \\
P & \xrightarrow{\;\alpha\;} & \mathbf{2}
\end{array}
$$

where P is primitive, there is a unique factorization $t : S/\sim_h \to P$ *such that* $t \circ \pi_h = \sigma$.

Proof. (1) It is clear that $\sim_h$ is reflexive and symmetric. For transitivity, suppose that $a \sim_h f \sim_h b$. Pick $c, c', d, d' \in S$ are such that $h(c) = 0 = h(c')$, $h(d) = 1 = h(d')$, $(a \wedge d) \vee c = (f \wedge d) \vee c$, and $(b \wedge d') \vee c' = (f \wedge d') \vee c'$. Put $c'' := c \vee c'$ and $d'' := d \wedge d'$, then $h(c'') = 0$ and $h(d'') = 1$ since h is a homomorphism. Note that the elements $(a \wedge d'') \vee c''$ and $(b \wedge d'') \vee c''$ are in the same $\mathcal{D}$-class, and that both are below $f \vee c''$. Therefore, by Lemma 5.1.7.2, $(a \wedge d'') \vee c'' = (b \wedge d'') \vee c''$, and we obtain $a \sim_h b$. We now show that $\sim_h$ is a congruence for the operations $\vee$ and $\wedge$. Suppose that $a \sim_h a'$, and let $b \in S$. We prove that $a \vee b \sim_h a' \vee b$ and $b \vee a \sim_h b \vee a'$. Pick $c, d \in S$ such that $h(c) = 0$, $h(d) = 1$ and $(a \wedge d) \vee c = (a' \wedge d) \vee c$. To prove that $((a \vee b) \wedge d) \vee c = ((a' \vee b) \wedge d) \vee c$, we use distributivity and left-handedness, as follows:

$$
\begin{aligned}
((a \vee b) \wedge d) \vee c &= (a \wedge d) \vee (b \wedge d) \vee c & \text{(strong distributivity)} \\
&= (a \wedge d) \vee c \vee (b \wedge d) \vee c & \text{(left-handedness)} \\
&= (a' \wedge d) \vee c \vee (b \wedge d) \vee c & \text{(assumption)} \\
&= (a' \wedge d) \vee (b \wedge d) \vee c & \text{(left-handedness)} \\
&= ((a' \vee b) \wedge d) \vee c. & \text{(strong distributivity)}
\end{aligned}
$$

By a similar, but slightly simpler, calculation, $(b \vee a) \wedge d = (b \vee a') \wedge d$. The proof that $\sim_h$ is also a congruence for the operation $\wedge$ on both sides proceeds along similar lines, using left normality (Lemma 5.1.7.1), and is left for the reader to verify. To see that $\sim_h$ refines $\ker(h)$, suppose that $a \sim_h b$. Pick $c, d \in S$ as in the definition of $\sim_h$. Then

$$
h(a) = (h(a) \wedge h(d)) \vee h(c) = h((a \wedge d) \vee c) = h((b \wedge d) \vee c)) = h(b),
$$

so indeed $(a, b) \in \ker(h)$.

(2) We need to show that, for any $a, b \in S$, $[a]_{\sim_h} \mathcal{D} [b]_{\sim_h}$ if, and only if, $h(a) = h(b)$. Note that if $[a]_{\sim_h} \mathcal{D} [b]_{\sim_h}$, then $h(a)$ is also $\mathcal{D}$-equivalent to $h(b)$ in **2**, which means that $h(a) = h(b)$. For the converse, suppose that

$h(a) = h(b)$. We distinguish the two cases (i) $h(a) = 1 = h(b)$ and (ii) $h(a) = 0 = h(b)$. In case (i), we get $[a \wedge b]_{\sim_h} = [a]_{\sim_h}$ by choosing $c := 0$ and $d := b \wedge a$ as in the definition of $\sim_h$. Similarly, $[b \wedge a]_{\sim_h} = [a]_{\sim_h}$. Hence, $\pi_h(a)$ and $\pi_h(b)$ are $\mathcal{D}$-equivalent. In case (ii), since h is proper, pick some $d \in S$ such that $h(d) = 1$. Then $(a \wedge d) \vee a = a = (0 \wedge d) \vee a$, so we get that $a \sim_h 0$. Similarly, $b \sim_h 0$, so in fact we obtain $[a]_{\sim_h} = [0]_{\sim_h} = [b]_{\sim_h}$. In particular, $[a]_{\sim_h} \mathcal{D} [b]_{\sim_h}$. We conclude that the $\mathcal{D}$-class of $[0]_{\sim_h}$ is $h^{-1}(0)$, and that, for any a with $h(a) = 1$, the $\mathcal{D}$-class of $[a]_{\sim_h}$ is $h^{-1}(1)$.

(3) Suppose that $\sigma : S \twoheadrightarrow P$ is a primitive quotient of S such that $\alpha \circ \sigma = h$. It is clear that there is at most one factorization of π_h through σ. We now show that the assignment $[a]_{\sim_h} \mapsto \sigma(a)$ does not depend on the choice of representative for the class $[a]_{\sim_h}$. Suppose that $a \sim_h a'$. If $h(a) = 0 = h(a')$, then $[\sigma(a)]_{\mathcal{D}} = h(a) = 0$, so $\sigma(a) = 0$, and similarly $\sigma(a') = 0$. Otherwise, we have $h(a) = 1 = h(a')$. Pick $c, d \in S$ such that $h(c) = 0$, $h(d) = 1$ and $(a \wedge d) \vee c = (a' \wedge d) \vee c$. As before, since $h(c) = 0$, we have $\sigma(c) = 0$. Since P is primitive, we have, for any non-zero $x, y \in P$, that $x \wedge y = x$. Hence

$$\sigma(a) = \sigma(a) \wedge \sigma(d) = (\sigma(a) \wedge \sigma(d)) \vee \sigma(c) = \sigma((a \wedge d) \vee c),$$

and similarly $\sigma(a') = \sigma((a' \wedge d) \vee c)$. Since $(a \wedge d) \vee c = (a' \wedge d) \vee c$, we conclude that $\sigma(a) = \sigma(a')$. $\qquad\square$

Remark 5.4.5. In light of this proposition, more can be said about the structure of primitive quotients of a left-handed strongly distributive skew lattice S. There is a natural a partial order on the set of quotients of S, defined by saying that a quotient $q : S \to Q$ is *below* another quotient $q' : S \to Q'$ if the map q factors through q'. Suppose $p : S \to P$ is any primitive quotient of S. Then $h := \alpha \circ p : S \to \mathbf{2}$ is a minimal quotient of S below the primitive quotient P, and $S/\sim_h$ is a maximal primitive quotient of S which is above P. The partially ordered set of primitive quotients of S is thus partitioned, and each primitive quotient lies between a unique maximal and minimal primitive quotient of S. The minimal primitive quotients of S are exactly the elements of the base space X, and the non-zero elements of the maximal primitive quotients will be the elements of the étalé space $S_\star$, see below.

Remark 5.4.6. An alternative way to define the equivalence relation $\sim_h$ on S is the following. Let us call a subset F of S a *preprime filter over h* if it satisfies the following properties:

1. if $a \in F$, $b \in S$ and $a \leq b$, then $b \in F$;

2. if $a, b \in F$ then $a \wedge b \in F$;

3. if $a \in F$, $b \in S$ and $h(b) = 0$, then $a \vee b \in F$;

4. if $a \in F$, then $h(a) = 1$;

5. if $b \in S$ and $h(b) = 1$, then there is $a \in F$ such that $[a]_{\mathcal{D}} = [b]_{\mathcal{D}}$.

We call a preprime filter over h a *prime filter over h* if it is minimal among the preprime filters over h. One may then show that the non-zero equivalence classes in $S/\sim_h$ (viewed as subsets of S) are exactly the prime filters over h. Therefore, the equivalence relation $\sim_h$ can also be described as the equivalence relation inducing the partition whose classes are the prime filters over h, and $h^{-1}(0)$.

Let S be a left-handed strongly distributive skew lattice. Write X for the local Priestley space $(S/\mathcal{D})_*$ that is dual to the lattice reflection $S/\mathcal{D}$ of S. Note that, by definition of $(S/\mathcal{D})_*$ and Theorem 5.1.3, elements of X can be represented as proper homomorphisms $h : S \to \mathbf{2}$. We now define the étale map $q : S_\star \to X$ dual to the skew lattice S.

Definition 5.4.7. Let S and X be as above. For $h \in X$, we define *the stalk over h* to be the non-zero $\mathcal{D}$-class of $S/\sim_h$. The *set underlying the space $S_\star$* is defined as the disjoint union of the stalks, that is,

$$S_\star := \bigsqcup_{h \in X} (S/\sim_h)^1 = \{(h, [a]_{\sim_h}) \mid h \in X, h(a) = 1\}.$$

The function $q : S_\star \to X$ is defined by $q((h, [a]_{\sim_h}) := h$. Recall that each $a \in S$ defines a compact open downset $\hat{a} = \{h \in X \mid h(a) = 1\}$ of X. For any $a \in S$, we naturally define the local section $s_a : \hat{a} \to S_\star$ of q over $\hat{a}$ by $s_a(h) := (h, [a]_{\sim_h})$. Finally, define the topology on $S_\star$ by taking the sets $\mathrm{im}(s_a)$ as a subbasis for the open sets, where a ranges over S.

Lemma 5.4.8. *Each function $s_a : \hat{a} \to S_\star$ is continuous and $q : S_\star \twoheadrightarrow X$ is a surjective étale map.*

Proof. Let $a, b \in S$ be arbitrary. We need to show that the set $s_a^{-1}(\mathrm{im}(s_b))$ is open in X. Let $h \in s_a^{-1}(\mathrm{im}(s_b))$ be arbitrary. Then $h \in \hat{a}$ and $a \sim_h b$. Pick $c, d \in S$ such that $h(c) = 0, h(d) = 1$ and $(a \wedge d) \vee c = (b \wedge d) \vee c$, by definition of $\sim_h$. Let $U_h := (\hat{d} \setminus \hat{c}) \cap \hat{a} \cap \hat{b}$. Note that $h \in U_h$, since $h(b) = h(a) = 1$. Moreover, for any $h' \in U_h$, we also have $h'(a) = 1 = h'(b), h'(c) = 0$ and $h'(d) = 1$, so that $a \sim_{h'} b$. We conclude that $h \in U_h \subseteq s_a^{-1}(\mathrm{im}(s_b))$, so $s_a^{-1}(\mathrm{im}(s_b))$ is open. Since each $h \in X$ is proper, each stalk $(S/\sim_h)^1$ is non-empty, so q is surjective. To prove that q is an étale map, let $e = (h, [a]_{\sim_h})$, which is in $S_\star$. Then $q|_{\mathrm{im}(s_a)} : \mathrm{im}(s_a) \to \hat{a}$ has s_a as its continuous inverse. $\square$

5.5. Proof of the duality theorem

In this section, we will prove that the contravariant functor $(-)^\star$ from $\mathbf{Sh}(\mathbf{LPS})$ to $\mathbf{SDL}$ (cf. Proposition 5.3.3 above) is full, faithful and essentially surjective. By a basic result from category theory (cf., e.g., [99, Thm IV.4.1]) it then follows that $(-)^\star$ is part of a dual equivalence of categories, proving Theorem 5.2.6. The proof that $(-)^\star$ is full and faithful is reasonably straightforward.

Proposition 5.5.1. *The contravariant functor $(-)^\star$ is full and faithful.*

Proof. Let E and F be sheaves over local Priestley spaces X and Y, respectively. We show that the assignment which sends a $\mathbf{Sh}(\mathbf{LPS})$-morphism $(f, \lambda) : (X, E) \to (Y, F)$ to the $\mathbf{SDL}$-morphism $(f, \lambda)^\star : (Y, F)^\star \to (X, E)^\star$ is a bijection. If $(f, \lambda)^\star = (g, \mu)^\star$, then $f^* = g^*$, using Lemma 5.3.2. Therefore, by Priestley duality, $f = g$. Moreover, if U is a basic open set in Y, then $\lambda_U = \mu_U$, using the definition of $(f, \lambda)^\star = (g, \mu)^\star$. Since a natural transformation between sheaves is entirely determined by its action on a basis of open sets (Lemma 5.2.5), it follows that $\lambda = \mu$. This concludes the proof that $(-)^\star$ is faithful. If $h : (Y, F)^\star \to (X, E)^\star$ is a homomorphism of skew lattices, then $\overline{h}$ is a proper homomorphism, so by Priestley duality, there is a unique $f : X \to Y$ such that $\overline{h} = \mathcal{L}(f) = f^{-1}$. For U a basic open, define $\lambda_U : F(U) \to E(f^{-1}(U))$ by sending $s \in F(U)$ to $h(s)$, which is indeed an element of $E(\overline{h}(s)) = E(f^{-1}(U))$. Now, if U is an arbitrary open and $s \in F(U)$, we can write U as a union of basic open sets $(U_i)_{i \in I}$. Then also $f^{-1}(U)$ is the union of the basic open sets $(f^{-1}(U_i))_{i \in I}$. It follows from the fact that h is a homomorphism that $(h(s)|_{f^{-1}(U_i)})_{i \in I}$ is a compatible family, so there is a unique patch in $E(f^{-1}(U))$, which we define to be $\lambda_U(s)$. We leave it to the reader to check that λ is a natural transformation and that $(f, \lambda)^\star = h$. $\qquad\square$

The proof that $(-)^\star$ is essentially surjective is more involved, and uses the construction of $S_\star$ from the previous section. Throughout the rest of this section, let S be a left-handed strongly distributive skew lattice. By Lemma 5.4.8, we have an étalé space $S_\star$ over the local Priestley space X, which is by definition $\mathbf{SDL}(S, \mathbf{2})$. The algebra $(S_\star)^\star$ is the skew lattice of local sections of $S_\star$ with compact open downward closed domains. There is a natural function $\varphi : S \to (S_\star)^\star$ which sends $a \in S$ to $s_a \in (S_\star)^\star$ (cf. Lemma 5.4.8). We will show in Propositions 5.5.2, 5.5.4 and 5.5.6 that φ is an isomorphism of skew lattices.

Proposition 5.5.2. *The function $\varphi : S \to (S_\star)^\star$ is a homomorphism of skew lattices.*

Proof. It is clear that φ preserves 0. Let $a, b \in S$. We need to show that $s_{a \vee b} = s_a \vee s_b$ and $s_{a \wedge b} = s_a \wedge s_b$. Note that in these equations, the operations $\vee$ and $\wedge$ on the right hand side are the operations defined in (5.3) and (5.4) of Section 5.3, whereas the operations $\vee$ and $\wedge$ on the left hand side are the operations of the given left-handed strongly distributive skew lattice S. Note that the domain of $s_{a \vee b}$ is $\widehat{a \vee b} = \widehat{a} \cup \widehat{b}$, which is also the domain of $s_a \vee s_b$. We now claim that $s_{a \vee b}(x) = s_b(x)$ for $x \in \widehat{b}$, and that $s_{a \vee b}(x) = s_a(x)$ for $x \in \widehat{a} \setminus \widehat{b}$, agreeing with the definition of $s_a \vee s_b$.

- Let $x \in \widehat{b}$. For $d := b$ and $c := 0$, it is easy to show that
  $((a \vee b) \wedge d) \vee c = (b \wedge d) \vee c$, so $[a \vee b]_{\sim_x} = [b]_{\sim_x}$.

- Let $x \in \widehat{a} \setminus \widehat{b}$. For $d := a$ and $c := b$, we then have
  $((a \vee b) \wedge d) \vee c = (a \wedge d) \vee c$, so that $[a \vee b]_{\sim_x} = [a]_{\sim_x}$.

Similarly, the domain of $s_{a \wedge b}$ is equal to the domain of $s_a \wedge s_b$, and if x is an element of this domain, then we have $((a \wedge b) \wedge d) \vee c = (a \wedge d) \vee c$, for $d := a \wedge b$ and $c := 0$, proving that $[a \wedge b]_{\sim_x} = [a]_{\sim_x}$. $\square$

To establish surjectivity of φ, we will need the following lemma.

Lemma 5.5.3. *For each $n \in \mathbb{N}$, the following holds. If $s : U \to S_\star$ is a section on a compact open downward closed subset U of X, and if $a_1, \ldots, a_n, c_1, \ldots, c_n, d_1, \ldots, d_n$ are elements of S such that*

1. *for each $i \in \{1, \ldots, n\}$, $\widehat{c}_i \subseteq \widehat{d}_i$;*

2. *$U = \bigcup_{i=1}^{n}(\widehat{d}_i \setminus \widehat{c}_i)$;*

3. *for each $i \in \{1, \ldots, n\}$, $\widehat{d}_i \setminus \widehat{c}_i \subseteq \widehat{a}_i$, and $s|_{\widehat{d}_i \setminus \widehat{c}_i} = s_{a_i}|_{\widehat{d}_i \setminus \widehat{c}_i}$,*

then there exists an element $a \in S$ such that $s = s_a$.

Proof. By induction on $n \in \mathbb{N}$. For $n = 0$, it follows from assumption (2) that $U = \emptyset$, so s is the empty function, and the (unique) element of S such that $s = s_a$ is $a = 0$. Let $n \geq 1$, and assume the statement is true for $n - 1$. Suppose that $s : U \to S_\star$, $a_1, \ldots, a_n, c_1, \ldots, c_n$, and $d_1, \ldots, d_n$ satisfy the assumptions (1)–(3).

Fix $j \in \{1, \ldots, n\}$. We will apply the induction hypothesis to the function $s|_{\widehat{c}_j} : \widehat{c}_j \to S_\star$. For $i \in \{1, \ldots, n\}$ with $i \neq j$, define $c_{i,j} := c_i \wedge c_j$ and $d_{i,j} := d_i \wedge c_j$. Note that

$$\widehat{d_{i,j}} \setminus \widehat{c_{i,j}} = (\widehat{d}_i \setminus \widehat{c}_i) \cap \widehat{c}_j,$$

so that $\widehat{c}_j = U \cap \widehat{c}_j = \bigcup_{i \neq j}(\widehat{d_{i,j}} \setminus \widehat{c_{i,j}})$. Assumptions (1) and (3) also clearly hold for the elements $a_i, c_{i,j}, d_{i,j}$, where i ranges over $\{1, \ldots, n\} \setminus \{j\}$. By the induction hypothesis, there exists an $f_j \in S$ such that $s|_{\widehat{c}_j} = s_{f_j}$.

Thus, for each j there exists an $f_j \in S$ such that $s|_{\widehat{c}_j} = s_{f_j}$. Consider the element

$$a := \bigvee_{j=1}^{n} ((a_j \wedge d_j) \vee f_j).$$

We claim that $s = s_a$. Note first that

$$\text{dom}(s_a) = \widehat{a} = \bigcup_{j=1}^{n} ((\widehat{a}_j \cap \widehat{d}_j) \cup \widehat{f}_j) = \bigcup_{j=1}^{n} \widehat{d}_j = U.$$

Now let $x \in U$ be arbitrary, and let j be the largest number in $\{1, \dots, n\}$ such that $x \in \widehat{d}_j$. Using Proposition 5.5.2 and the definition of $\vee$ in $(S_\star)^\star$, we see that

$$s_a(x) = \begin{cases} s_{f_j}(x) & \text{if } x \in \widehat{c}_j, \\ s_{a_j}(x) & \text{if } x \notin \widehat{c}_j. \end{cases}$$

If $x \in \widehat{c}_j$, then $s_{f_j}(x) = s(x)$ by the choice of f_j, and if $x \notin \widehat{c}_j$, then $x \in \widehat{d}_j \setminus \widehat{c}_j$, so $s_{a_j}(x) = s(x)$ by assumption (3). $\qquad\square$

The above lemma exactly enables us to prove surjectivity of φ: it is now an application of compactness, as follows.

Proposition 5.5.4. *The function* $\varphi : S \to (S_\star)^\star$ *is surjective, and in particular it is a morphism of* **SDL**.

Proof. Let $s \in (S_\star)^\star$, so s is a continuous section over a compact open downset U. For each $x \in U$, we have $s(x) \in (S_\star)_x = (S/{\sim_x})^1$, so we can pick $a_x \in S$ such that $s(x) = [a_x]_{\sim_x}$, and define

$$T_x := \|s = s_{a_x}\| = \{y \in U \mid s(y) = [a_x]_{\sim_y}\} = s^{-1}(\text{im}(s_{a_x})) \cap U.$$

Note that T_x is open in X, because s is continuous, $\text{im}(s_{a_x})$ is open in $S_\star$, and U is open in X. Since $x \in T_x$, there exist $c_x, d_x \in S$ such that $x \in \widehat{d}_x \setminus \widehat{c}_x \subseteq T_x$, where we may assume without loss of generality that $\widehat{c}_x \subseteq \widehat{d}_x$. We now have

$$U \subseteq \bigcup_{x \in U} (\widehat{d}_x \setminus \widehat{c}_x) \subseteq \bigcup_{x \in U} T_x \subseteq U,$$

so equality holds throughout. Since U is compact, there exist elements $x_1, \dots, x_n \in U$ such that $U = \bigcup_{i=1}^{n} (\widehat{d}_{x_i} \setminus \widehat{c}_{x_i})$. We will write c_i and d_i for c_{x_i} and d_{x_i}, respectively. Note that, for each $i \in \{1, \dots, n\}$, we have $\widehat{d}_i \setminus \widehat{c}_i \subseteq T_{x_i}$, so $s|_{\widehat{d}_i \setminus \widehat{c}_i} = s_{a_{x_i}}|_{\widehat{d}_i \setminus \widehat{c}_i}$. By Lemma 5.5.3, we get $a \in S$ such that $s = s_a$, proving that φ is surjective. For the 'in particular' part, note that surjective homomorphisms are always proper. $\qquad\square$

Lemma 5.5.5. *For each $n \in \mathbb{N}$, the following holds. If $a, b, c_1, \ldots, c_n$ and $d_1, \ldots d_n$ are elements of S such that:*

1. *for each $i \in \{1, \ldots, n\}$, $\widehat{c_i} \subseteq \widehat{d_i} \subseteq \widehat{a}$;*

2. *$\widehat{a} = \bigcup_{i=1}^{n} (\widehat{d_i} \setminus \widehat{c_i}) = \widehat{b}$;*

3. *for each $i \in \{1, \ldots, n\}$, $(a \wedge d_i) \vee c_i = (b \wedge d_i) \vee c_i$,*

then $a = b$.

Proof. For $n = 0$, we get that $\widehat{a} = \varnothing = \widehat{b}$, so $a = 0 = b$. Let $n \geq 1$, and suppose the statement is proved for $n - 1$. Let $c_1, \ldots, c_n, d_1, \ldots d_n$ be elements of S satisfying the assumptions. Then in particular $\widehat{a} = \bigcup_{i=1}^{n} \widehat{d_i} = \widehat{\bigvee_{i=1}^{n} d_i}$, so that $[a]_{\mathcal{D}} = [\bigvee_{i=1}^{n} d_i]_{\mathcal{D}}$. Therefore,

$$a = a \wedge \left(\bigvee_{i=1}^{n} d_i \right) = \bigvee_{i=1}^{n} (a \wedge d_i).$$

Similarly, since $\widehat{b} = \bigcup_{i=1}^{n} \widehat{d_i}$, we get that $b = \bigvee_{i=1}^{n} (b \wedge d_i)$.

Let $j \in \{1, \ldots, n\}$ be arbitrary. For $i \neq j$, define $a_j := a \wedge c_j$, $b_j := b \wedge c_j$, $d_{i,j} := d_i \wedge c_j$, and $c_{i,j} := c_i \wedge c_j$. Note that $\widehat{a_j} = \widehat{b_j}$, and also that for each $i \neq j$, we have $\widehat{c_{i,j}} \subseteq \widehat{d_{i,j}} \subseteq \widehat{a_j}$. Moreover:

$$
\begin{aligned}
(a_j \wedge d_{i,j}) \vee c_{i,j} &= (a \wedge c_j \wedge d_i \wedge c_j) \vee (c_i \wedge c_j) && \text{(definitions of } a_j, d_{i,j} \text{ and } c_{i,j}) \\
&= (a \wedge d_i \wedge c_j) \vee (c_i \wedge c_j) && \text{(left normality)} \\
&= ((a \wedge d_i) \vee c_i) \wedge c_j && \text{(strong distributivity)} \\
&= ((b \wedge d_i) \vee c_i) \wedge c_j && \text{(assumption)} \\
&= (b \wedge d_i \wedge c_j) \vee (c_i \wedge c_j) && \text{(as above)} \\
&= (b_j \wedge d_{i,j}) \vee c_{i,j}.
\end{aligned}
$$

By the induction hypothesis, we thus conclude that $a \wedge c_j = a_j = b_j = b \wedge c_j$. Now, to show $a \wedge d_j = b \wedge d_j$, we calculate:

$$
\begin{aligned}
a \wedge d_j &= a \wedge (d_j \vee c_j) && (\widehat{c_j} \subseteq \widehat{d_j}) \\
&= (a \wedge d_j) \vee (a \wedge c_j) && \text{(strong distributivity)} \\
&= (a \wedge d_j) \vee (b \wedge c_j) && (a \wedge c_j = b \wedge c_j) \\
&= (a \wedge d_j) \vee c_j \vee (b \wedge c_j) && (\widehat{c_j} \subseteq \widehat{b}) \\
&= (b \wedge d_j) \vee c_j \vee (b \wedge c_j) && \text{(assumption)} \\
&= (b \wedge d_j) \vee (b \wedge c_j) && (\widehat{c_j} \subseteq \widehat{b}) \\
&= b \wedge d_j. && \text{(as above)}
\end{aligned}
$$

Now $a = \bigvee_{j=1}^n (a \wedge d_j) = \bigvee_{j=1}^n (b \wedge d_j) = b$, as required.   □

Proposition 5.5.6. *The function $\varphi : S \to (S_\star)^\star$ is injective.*

Proof. Let $a, b \in S$, and suppose that $s_a = s_b$. We then in particular have that $\hat{a} = \mathrm{dom}(s_a) = \mathrm{dom}(s_b) = \hat{b}$. For each $x \in \hat{a}$, $[a]_{\sim_x} = s_a(x) = s_b(x) = [b]_{\sim_x}$, so by definition of $\sim_x$, pick $c_x, d_x \in S$ such that $(a \wedge d_x) \vee c_x = (b \wedge d_x) \vee c_x$, and $x \in \hat{d_x} \setminus \hat{c_x}$. We thus get that the collection $(\hat{d_x} \setminus \hat{c_x})_{x \in \hat{a}}$ is an open cover of $\hat{a}$. Since $\hat{a}$ is compact, we can pick a finite subcover, indexed by $x_1, \ldots, x_n \in \hat{a}$. We will write c_i and d_i for c_{x_i} and d_{x_i}, respectively. Without loss of generality, we may assume that $\hat{c_i} \subseteq \hat{d_i} \subseteq \hat{a}$ for each i, by replacing c_i by $c_i \wedge d_i \wedge a$ and d_i by $d_i \wedge a$, and checking that the new c_i and d_i still satisfy the same properties. Now it follows from Lemma 5.5.5 that $a = b$.   □

We have thus established that $\varphi : S \to (S_\star)^\star$ is an isomorphism in **SDL**, so:

Proposition 5.5.7. *The contravariant functor $(-)^\star : \mathbf{Sh}(\mathbf{LPS}) \to \mathbf{SDL}$ is essentially surjective.*

It now follows from Propositions 5.5.1 and 5.5.7 that $(-)^\star$ is part of a dual equivalence. This concludes the proof of our main theorem, Theorem 5.2.6.

Concluding remarks

It is a central fact in logic that every distributive lattice has a free Boolean extension, or Booleanization, cf. Example 1.1.13 in Chapter 1. This fact is at the base of the relationship between intuitionistic and Boolean logic. It would be interesting to seek a non-commutative counterpart of this result. Since the classical result is most transparently understood via duality, it is likely that our duality would prove useful. Furthermore, a "skew Heyting algebra" is a notion still needing to be properly defined. In a recent paper [51] on Esakia's work, Gehrke showed that Heyting algebras may be understood as those distributive lattices for which the embedding in their Booleanization has a right adjoint. This could provide a natural starting point for the exploration of skew Heyting algebras.

A different natural non-commutative generalization of distributive lattices is a class of inverse semigroups whose idempotents form a distributive lattice. Recently, Stone duality has been generalized to this setting [101, 102, 103]. The most recent work in this direction [103] generalizes Stone's duality between distributive lattices and spectral spaces to the context of inverse semigroups. However, to the best of our knowledge, Priestley's duality for distributive lattices has not yet been generalized to inverse semigroups. The results in this chapter might also be fruitfully applied to obtain such a duality for a class of inverse semigroups. We leave this as an interesting direction for future work.

Chapter 6. Distributive envelopes and topological duality for lattices

In this chapter, we establish a topological duality for bounded lattices. The two main features of our duality are that it generalizes Stone duality for bounded distributive lattices, and that the morphisms on either side are not the standard ones. A positive consequence of the choice of morphisms is that those on the topological side are functional. Towards obtaining the topological duality, we develop a universal construction which associates to an arbitrary lattice two distributive lattice envelopes with a Galois connection between them. This is a modification of a construction of the injective hull of a semilattice by Bruns and Lakser, adjusting their concept of 'admissibility' to the finitary case. Finally, we show that the dual spaces of the distributive envelopes of a lattice coincide with completions of quasi-uniform spaces naturally associated with the lattice, thus giving a precise spatial meaning to the distributive envelopes. This chapter is a modified version of the paper [64].

Topological duality for Boolean algebras [137] and distributive lattices [138] is a useful tool for studying relational semantics for propositional logics. Canonical extensions [84, 85, 56, 54] provide a way of looking at these semantics algebraically. In the absence of a satisfactory topological duality, canonical extensions have been used [40] to treat relational semantics for logics with lattice-based algebraic semantics. The relationship between canonical extensions and topological dualities in the distributive case suggests that canonical extensions should be taken into account when looking for a topological duality for arbitrary bounded[1] lattices. The main aim of this chapter is to investigate this line of research.

Several different approaches to topological duality for lattices exist in the literature, starting from Urquhart [139]. Important contributions were made, among others, by Hartung [78, 79], who connected Urquhart's duality to the Formal Concept Analysis [48] approach to lattices. However, as we will show in Section 6.2 of this chapter, a space which occurs in Hartung's duality can be rather ill-behaved. In particular, such a space need not be sober, and therefore it need not occur as the Stone dual space of any distributive lattice. By contrast, the spaces that occur in the duality developed in this chapter are Stone dual spaces of certain distributive lattices that are naturally associated to the given lattice. In topological duality for lattices

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[1]From here on, we will adopt the convention that all lattices considered in this chapter are bounded.

[79], the morphisms in the dual category are necessarily relational rather than functional. In this chapter, we exhibit a class of lattice morphisms, the 'admissible homomorphisms', for which the morphisms in the dual category can still be functional. However, the topological charaterization of the dual category is still rather involved. In the last section of this chapter, we therefore propose a different spatial approach to lattices in the form of quasi-uniform spaces.

Outline of the chapter. We first develop a relevant piece of order theory that may be of independent interest. The ideas that play a role here originate with the construction of the injective hull of a semilattice [19], which is a frame. In Section 6.1, we recast this construction in the finitary setting to obtain a construction of a pair of distributive lattices from a given lattice, which we shall call the *distributive envelopes* of the lattice. Moreover, as we will also see in Section 6.1, these two distributive envelopes correspond to the meet- and join-semilattice-reducts of the lattice of departure and are linked by a Galois connection whose lattice of Galois-closed sets is isomorphic to the original lattice. In Section 6.2, we then use Stone-Priestley duality for distributive lattices [138, 127] and the theory of canonical extensions to find an appropriate category dual to the category of lattices, using this representation of a lattice as a pair of distributive lattices with a Galois connection between them. Particular attention is devoted to morphisms; the algebraic results from Section 6.1 will guide us towards a notion of 'admissible morphism' between lattices, which have the property that their topological duals are functional. Finally, in Section 6.3, we will propose quasi-uniform spaces as an alternative to topology for studying set representations of lattices.

6.1. Distributive envelopes

In this section we introduce the *distributive envelopes* $D^{\wedge}(L)$ and $D^{\vee}(L)$ of a lattice L. After giving the universal property that defines the envelopes, we will give both a point-free and a point-set construction of it, and investigate its categorical properties. A lattice has two semilattice reducts, each of which gives rise to a corresponding distributive envelope. These two distributive envelopes of a lattice are linked by a Galois connection, which enables one to recover the original lattice L. Some of the results in this section can be seen as finitary versions of the results on injective hulls of semilattices of Bruns and Lakser [19]. We will relate our results to theirs in Remark 6.1.15. However, the reader who is not familiar with [19] should be able to read this section independently. The following definition is central; it is the finitary version of 'admissibility' in [19].

Definition 6.1.1. Let L be a lattice. A finite subset $M \subseteq L$ is *join-admissible* if its join distributes over all meets with elements from L, i.e., if, for all $a \in L$,

$$a \wedge \bigvee M = \bigvee_{m \in M} (a \wedge m).$$

We say that a function $f : L_1 \to L_2$ between lattices *preserves admissible joins* if, for each finite join-admissible set $M \subseteq L_1$, we have $f(\bigvee M) = \bigvee_{m \in M} f(m)$.

Note that a lattice is distributive if, and only if, all finite subsets are join-admissible.

Definition 6.1.2. Let L be a lattice. An embedding $\eta_L^\wedge : L \hookrightarrow D^\wedge(L)$ of L into a distributive lattice $D^\wedge(L)$ which preserves finite meets and admissible joins is called a *distributive $\wedge$-envelope* of L if it satisfies the following universal property:

For any function $f : L \to D$ into a distributive lattice D that preserves

finite meets and admissible joins, there exists a unique lattice homomorphism

$\widehat{f} : D^\wedge(L) \to D$ such that $\widehat{f} \circ \eta_L^\wedge = f$, i.e., the following diagram commutes:

$$
\begin{array}{ccc}
L & \xrightarrow{\;\eta_L^\wedge\;} & D^\wedge(L) \\
 & {}_{f}\searrow & \Big\downarrow{\scriptstyle !}\,\widehat{f} \\
 & & D
\end{array}
$$

The definition of the *distributive $\vee$-envelope*, $D^\vee(L)$, of L is order dual, cf. Remark 6.1.16.

Let us give some intuition for the above definitions. The join-admissible subsets of L are those subsets whose joins 'are already distributive' in L. A distributive $\wedge$-envelope of a lattice L is a universal solution to the question of embedding L as a $\wedge$-semilattice into a distributive lattice while preserving all admissible joins. Note that a join that is not admissible can not be preserved by any $\wedge$-embedding into a distributive lattice; in this sense, a distributive $\wedge$-envelope 'adds as few joins as possible' to make L distributive.

The main aim of this section is to show that the distributive $\wedge$-envelope of a lattice always exists (Theorem 6.1.4); it is clearly unique up to isomorphism. The same results of course hold for the distributive $\vee$-envelope. In proving these theorems, two different representations of $D^\wedge(L)$ will be useful, one is point-free, the other uses the set of 'points' $J^\infty(L^\delta)$ of the canonical extension of L. We first give the point-free construction of the distributive $\wedge$-envelope $D^\wedge(L)$ of L. To construct $D^\wedge(L)$, we want to 'add joins' to L.

This can of course be done with ideals. In the case of $D^\wedge(L)$ the required ideals will be closed under admissible joins. We thus define "a-ideals" as follows.

Definition 6.1.3. A subset $I \subseteq L$ is called an *a-ideal*[2] if (i) I is a downset, i.e., if $a \in I$ and $b \leq a$ then $b \in I$, and (ii) I is closed under admissible joins, i.e., if $M \subseteq I$ is join-admissible, then $\bigvee I \in A$.

Note that any lattice ideal is in particular an a-ideal. In the case of a distributive lattice, the lattice ideals and a-ideals coincide. Moreover, the set of all a-ideals of a lattice is a closure system: any intersection of a-ideals is again an a-ideal. Therefore, for any subset T of a lattice, there exists a smallest a-ideal containing T. We will denote this a-ideal by $\langle T \rangle_{ai}$ and call it *the a-ideal generated by T*. As usual, we say that an a-ideal I is *finitely generated* if there is a finite set T such that $I = \langle T \rangle_{ai}$. If L is a lattice, we denote by $\eta_L^\wedge$ the map which sends a lattice element $a \in L$ to the finitely generated a-ideal $\langle a \rangle_{ai}$, which coincides with the downset generated by a. We now set out to prove the following.

Theorem 6.1.4. *Let L be a lattice. The map $\eta_L^\wedge$ from L to the partially ordered set of finitely generated a-ideals of L is a distributive $\wedge$-envelope of L.*

This theorem can be proved by adopting the proof of [19, Theorem 2]. We will give an alternative proof that uses the canonical extension. The plan of our proof is as follows. In Lemmas 6.1.5 and 6.1.6, we characterize join-admissibility and finitely generated a-ideals using the canonical extension. Using these lemmas, we show that the finitely generated a-ideals form a distributive lattice (Proposition 6.1.7), and then prove Theorem 6.1.4. We refer to Section 1.2 in Chapter 1 for preliminaries on canonical extensions. Recall that in the canonical extension of a distributive lattice, completely join-irreducible elements are completely join-prime. The following lemma shows that, in the canonical extension of any lattice, a set is join-admissible if, and only if, the completely join-irreducibles still behave as completely join-primes with respect to the join of that set.

Lemma 6.1.5. *Let L be a lattice and $M \subseteq L$ a finite subset. The following are equivalent:*

1. *The set M is join-admissible;*

2. *For any $x \in J^\infty(L^\delta)$, if $x \leq \bigvee M$, then $x \leq m$ for some $m \in M$.*

---

[2]As an anonymous referee pointed out, this definition is a special case of a $\mathcal{Z}$-*join ideal* in the sense of, e.g., [44]. It would be interesting to see how the results in this section relate to those in [44].

Proof. For (1) $\Rightarrow$ (2), suppose that M is join-admissible, and let $x \in J^\infty(L^\delta)$ be such that $x \leq \bigvee M$. Define $x' := \bigvee_{m \in M}(x \wedge m)$. It is obvious that $x' \leq x$. We show that $x \leq x'$. Let y be an ideal element of L^δ such that $x' \leq y$. Then, for each $m \in M$, we have $x \wedge m \leq y$. By the compactness property of the canonical extension, there exists $a_m \in L$ such that $x \leq a_m$ and $a_m \wedge m \leq y$. Let $a := \bigwedge_{m \in M} a_m$. Since M is join-admissible, we get

$$x \leq a \wedge \bigvee M = \bigvee_{m \in M}(a \wedge m) \leq \bigvee_{m \in M}(a_m \wedge m) \leq y.$$

Since y was an arbitrary ideal element above x', by one of the equivalent properties of denseness ([54, Lemma 2.4]) we conclude that $x \leq x'$. So $x = x' = \bigvee_{m \in M}(x \wedge m)$. Since x is join-irreducible, we get $x = x \wedge m$ for some $m \in M$, so $x \leq m$. For (2) $\Rightarrow$ (1), let $a \in L$ be arbitrary. We only need to show that $a \wedge \bigvee M \leq \bigvee_{m \in M}(a \wedge m)$ holds in L; the other inequality is obvious. We will show that the inequality holds in L^δ and use that $L \hookrightarrow L^\delta$ is an embedding. Let $x \in J^\infty(L^\delta)$ such that $x \leq a \wedge \bigvee M$. By (2), pick $m \in M$ such that $x \leq m$. Then $x \leq a \wedge m$, which is below $\bigvee_{m \in M}(a \wedge m)$. Since $x \in J^\infty(L^\delta)$ was arbitrary, we conclude that $a \wedge \bigvee M \leq \bigvee_{m \in M}(a \wedge m)$, using the fact that the canonical extension is $\bigvee$-generated by $J^\infty(L^\delta)$ (Proposition 1.2.4 in Chapter 1). $\square$

Lemma 6.1.5 will be our main tool in the proof of Theorem 6.1.4. It is a typical example of the usefulness of canonical extensions: one can formulate an algebraic property (join-admissibility) in a spatial manner (using the 'points', i.e., completely join-irreducibles, of the canonical extension). Note that the proof of Lemma 6.1.5 goes through without the restriction that M is finite, if one extends the definition of join-admissibility to include infinite sets. We will not expand on this point here, because we will only need the result for finite sets, but we merely note that this observation can be used to give an alternative proof of the results in [19].

For any $a \in L$, we define $\widehat{a} := \{x \in J^\infty(L^\delta) : x \leq a\}$. Note that $\widehat{a \wedge b} = \widehat{a} \cap \widehat{b}$, for any $a, b \in L$. Lemma 6.1.5 says that M is join-admissible if, and only if, $\widehat{\bigvee M} = \bigcup_{m \in M} \widehat{m}$. We can use these observations to obtain the following characterization of the a-ideal generated by a finite subset.

Lemma 6.1.6. *Let L be a lattice, $T \subseteq L$ a finite subset and $b \in L$. The following are equivalent:*

1. *$b \in \langle T \rangle_{ai}$;*

2. *$\widehat{b} \subseteq \bigcup_{a \in T} \widehat{a}$;*

3. *There exists a finite join-admissible set $M \subseteq \downarrow T$ such that $b = \bigvee M$.*

Proof. For (1) implies (2), note that $I := \{b \in L : \widehat{b} \subseteq \bigcup_{a \in T} \widehat{a}\}$ is an a-ideal that contains T: it is clearly a downset, and it is closed under admissible joins, using Lemma 6.1.5. Therefore, $\langle T \rangle_{ai} \subseteq I$. For (2) implies (3), let $M := \{b \wedge a \mid a \in T\}$. We will show that $b = \bigvee M$ and M is join-admissible. Note that $\bigvee M \leq b$, so $\widehat{\bigvee M} \subseteq \widehat{b}$. Using (2), we also get:

$$\widehat{b} = \widehat{b} \cap \bigcup_{a \in T} \widehat{a} = \bigcup_{a \in T} (\widehat{b} \cap \widehat{a}) = \bigcup_{a \in T} \widehat{b \wedge a} = \bigcup_{m \in M} \widehat{m} \subseteq \widehat{\bigvee M} \subseteq \widehat{b}.$$

Therefore, equality holds throughout, and in particular we have that $b = \bigvee M$ and $\bigcup_{m \in M} \widehat{m} = \widehat{\bigvee M}$, so that M is join-admissible by Lemma 6.1.5. The direction (3) implies (1) is clear from the definition of a-ideal. $\qquad\square$

Now, towards proving Theorem 6.1.4, we first show that the finitely generated a-ideals form a distributive sublattice of the complete lattice of all a-ideals.

Proposition 6.1.7. *Let L be an arbitrary lattice, and let T and U be finite subsets of L. Then*

$$\langle T \rangle_{ai} \cap \langle U \rangle_{ai} = \langle t \wedge u \mid t \in T, u \in U \rangle_{ai}.$$

In particular, the intersection of two finitely generated a-ideals is again finitely generated, and the collection of finitely generated a-ideals of L forms a distributive lattice.

Proof. By Lemma 6.1.6, we have that

$$\langle T \rangle_{ai} \cap \langle U \rangle_{ai} = \{b \in L \mid \widehat{b} \subseteq \left(\bigcup_{t \in T} \widehat{t} \right) \cap \left(\bigcup_{u \in U} \widehat{u} \right) \}.$$

Note that

$$\left(\bigcup_{t \in T} \widehat{t} \right) \cap \left(\bigcup_{u \in U} \widehat{u} \right) = \bigcup_{t \in T, u \in U} (\widehat{t} \cap \widehat{u}) = \bigcup_{t \in T, u \in U} \widehat{t \wedge u}.$$

Hence, $\langle T \rangle_{ai} \cap \langle U \rangle_{ai} = \{b \in L \mid \widehat{b} \subseteq \bigcup_{t \in T, u \in U} \widehat{t \wedge u}\} = \langle t \wedge u \mid t \in T, u \in U \rangle_{ai}$, again by Lemma 6.1.6. Note that the set of generators for $\langle T \rangle_{ai} \cap \langle U \rangle_{ai}$ is in particular finite. To prove distributivity, let T, U_1, U_2 be finite subsets of L. Note that the a-ideal $\langle T \rangle_{ai} \cap (\langle U_1 \rangle_{ai} \vee \langle U_2 \rangle_{ai})$ is generated by $\{t \wedge u \mid t \in T, u \in U_1 \cup U_2\}$. Each of these generators is either in $\langle T \rangle_{ai} \cap \langle U_1 \rangle_{ai}$ or $\langle T \rangle_{ai} \cap \langle U_2 \rangle_{ai}$. Therefore, $\langle T \rangle_{ai} \cap (\langle U_1 \rangle_{ai} \vee \langle U_2 \rangle_{ai})$ is contained in $(\langle T \rangle_{ai} \cap \langle U_1 \rangle_{ai}) \vee (\langle T \rangle_{ai} \cap \langle U_2 \rangle_{ai})$, as required. $\qquad\square$

Proof of Theorem 6.1.4. We optimistically write $D^{\wedge}(L)$ for the distributive lattice of finitely generated a-ideals of L. Recall that $\eta_L^{\wedge}$ is the map which sends a to $\langle a \rangle_{ai} = \downarrow a$. It is clear that $\eta_L^{\wedge}$ is injective and preserves finite meets. Let M be a finite join-admissible set. Then $\bigvee_{m \in M} \eta_L^{\wedge}(m) = \langle M \rangle_{ai}$. By Lemma 6.1.6, we have $b \in \langle M \rangle_{ai}$ if, and only if, $\hat{b} \subseteq \bigcup_{m \in M} \widehat{m}$. By Lemma 6.1.5, we have $\bigcup_{m \in M} \widehat{m} = \widehat{\bigvee M}$ since M is join-admissible. We conclude that $\bigvee_{m \in M} \eta_L^{\wedge}(m) = \eta_L^{\wedge}(\bigvee M)$. Thus, $\eta_L^{\wedge}$ also preserves admissible joins. It remains to show that it satisfies the universal property. Let $f : L \to D$ be a function which preserves meets and admissible joins. If $g : D^{\wedge}(L) \to D$ is a homomorphism such that $g \circ \eta_L^{\wedge} = f$, then, for any finite subset $T \subseteq L$, we have

$$g(\langle T \rangle_{ai}) = g\left(\bigvee_{t \in T} \eta_L^{\wedge}(t) \right) = \bigvee_{t \in T} g(\eta_L^{\wedge}(t)) = \bigvee_{t \in T} f(t).$$

Hence, there is at most one homomorphism $g : D^{\wedge}(L) \to D$ satisfying $g \circ \eta_L^{\wedge} = f$. Let $\hat{f} : D^{\wedge}(L) \to D$ be the function defined for a finite subset $T \subseteq L$ by

$$\hat{f}(\langle T \rangle_{ai}) := \bigvee_{t \in T} f(t).$$

We show that $\hat{f}$ is a well-defined homomorphism. For well-definedness, suppose that $\langle T \rangle_{ai} = \langle U \rangle_{ai}$ for some finite subsets $T, U \subseteq L$. Let $u \in U$ be arbitrary. We then have $u \in \langle T \rangle_{ai}$. By Lemma 6.1.6, $u = \bigvee M$ for some finite join-admissible $M \subseteq \downarrow T$. Using that f preserves admissible joins and order, we get

$$f(u) = f\left(\bigvee M \right) = \bigvee_{m \in M} f(m) \leq \bigvee_{t \in T} f(t).$$

Since $u \in U$ was arbitrary, we have shown that $\bigvee_{u \in U} f(u) \leq \bigvee_{t \in T} f(t)$. The proof of the other inequality is the same. We conclude that $\bigvee_{t \in T} f(t) = \bigvee_{u \in U} f(u)$, so $\hat{f}$ is well-defined. It is clear that $\hat{f} \circ \eta_L^{\wedge} = f$. In particular, $\hat{f}$ preserves 0 and 1, since f does. It remains to show that $\hat{f}$ preserves $\vee$ and $\wedge$. Let $T, U \subseteq L$ be finite subsets. Since the a-ideal $\langle T \rangle_{ai} \vee \langle U \rangle_{ai}$ is generated by $T \cup U$, we get

$$\hat{f}(\langle T \rangle_{ai} \vee \langle U \rangle_{ai}) = \bigvee_{v \in T \cup U} f(v) = \bigvee_{t \in T} f(t) \vee \bigvee_{u \in U} f(u) = \hat{f}(\langle T \rangle_{ai}) \vee \hat{f}(\langle U \rangle_{ai}).$$

Using Proposition 6.1.7 and the assumptions that D is distributive and f is

meet-preserving, we have

$$\widehat{f}(\langle T\rangle_{ai} \wedge \langle U\rangle_{ai}) = \bigvee_{t\in T, u\in U} f(t \wedge u)$$

$$= \bigvee_{t\in T, u\in U} (f(t) \wedge f(u))$$

$$= \bigvee_{t\in T} f(t) \wedge \bigvee_{u\in U} f(u) = \widehat{f}(\langle T\rangle_{ai}) \wedge \widehat{f}(\langle U\rangle_{ai}). \qquad \square$$

In what follows, it will be useful to know that the unique lift of an injective map to the distributive envelope is still injective.

Proposition 6.1.8. *Let L be a lattice, D a distributive lattice, and $f : L \to D$ a function which preserves finite meets and admissible joins. If f is injective, then the unique extension $\widehat{f} : D^\wedge(L) \to D$ is injective.*

Proof. Note that f is order reflecting, since f is meet-preserving and injective. Suppose that $\widehat{f}(\langle U\rangle_{ai}) \le \widehat{f}(\langle T\rangle_{ai})$. We need to show that $\langle U\rangle_{ai} \subseteq \langle T\rangle_{ai}$. Let $u \in U$ be arbitrary. Then $f(u) \le \widehat{f}(\langle U\rangle_{ai}) \le \widehat{f}(\langle T\rangle_{ai}) = \bigvee_{t\in T} f(t)$. For any $a \in L$, we then have

$$f(a \wedge u) = f(a \wedge u) \wedge \bigvee_{t\in T} f(t) = \bigvee_{t\in T} (f(a \wedge u) \wedge f(t))$$

$$= \bigvee_{t\in T} f(a \wedge u \wedge t) \le f\left(\bigvee_{t\in T} (a \wedge u \wedge t)\right).$$

Since f is order reflecting, we thus get $a \wedge u \le \bigvee_{t\in T}(a \wedge u \wedge t)$. Since the other inequality is clear, we get

$$a \wedge u = \bigvee_{t\in T} (a \wedge u \wedge t). \qquad (6.1)$$

In particular, putting $a = 1$, we see that $u = \bigvee_{t\in T}(u \wedge t)$, and equation (6.1) then says that $\{u \wedge t \mid t \in T\}$ is join-admissible. Hence, $u \in \langle T\rangle_{ai}$. We conclude that $U \subseteq \langle T\rangle_{ai}$, and therefore $\langle U\rangle_{ai} \subseteq \langle T\rangle_{ai}$. $\qquad \square$

The following characterisation of the distributive envelope now follows easily.

Corollary 6.1.9. *Let L be a lattice. If D is a distributive lattice and $f : L \to D$ is an injective function that preserves meets and admissible joins, and the image of f join-generates D, then D is isomorphic to $D^\wedge(L)$ via the isomorphism $\widehat{f}$.*

Proof. The homomorphism $\widehat{f}$ is injective by Proposition 6.1.8. It is surjective because $f(L)$ is join-dense in D and $\widehat{f}(D^\wedge(L)) = \{\bigvee f(T) \mid T \subseteq L\}$, by the construction of $\widehat{f}$ in the proof of Theorem 6.1.4. $\qquad \square$

Using this corollary, we obtain the following alternative presentation of the distributive envelope $D^\wedge(L)$.

Corollary 6.1.10. *Let L be a lattice. Let φ be the function which sends a finitely generated a-ideal $I = \langle T \rangle_{ai}$ to the set $\bigcup_{a \in T} \widehat{a}$. Then φ is a well-defined isomorphism between $D^\wedge(L)$ and the sublattice of $\mathcal{P}(J^\infty(L^\delta))$ that is generated by the collection $\{\widehat{a} \mid a \in L\}$.*

Proof. Let f be the function that sends $a \in L$ to $\widehat{a} \in \mathcal{P}(J^\infty(L^\delta))$. By the remarks preceding Lemma 6.1.6, f preserves meets and admissible joins. By Proposition 1.2.4, f is injective. Now apply Corollary 6.1.9, noting that $\widehat{f} = \varphi$. □

We now investigate the categorical properties of the distributive $\wedge$-envelope. In particular, we will deduce that the assignment $L \mapsto D^\wedge(L)$ extends to an adjunction between categories. We first define the appropriate categories. We denote by **DL** the category of distributive lattices with homomorphisms. The relevant category of lattices is defined as follows.

Definition 6.1.11. We say that a function $f : L_1 \to L_2$ between lattices is a *join-admissible morphism* if f preserves finite meets and admissible joins, and, for any join-admissible set $M \subseteq L_1$, $f(M)$ is join-admissible. We denote by $\mathbf{L}_{a\vee}$ the category of lattices with join-admissible morphisms between them. (The reader may verify that $\mathbf{L}_{a\vee}$ is indeed a category.)

Note that if the lattice L_2 is distributive, then the condition that f sends join-admissible sets to join-admissible sets is vacuously true. This explains why we did not need to state this condition in the defining property of $D^\wedge(L)$. However, the following example shows that, in general, the condition in Definition 6.1.11 that f sends join-admissible sets to join-admissible sets can not be omitted.

Example 6.1.12. *It is not the case that if $f : L_1 \to L_2$ and $g : L_2 \to L_3$ preserve meets and admissible joins, then their composition gf preserves admissible joins.* Let L_1 be the Boolean algebra with 2 atoms, let L_2 be the three-element antichain with 0 and 1 adjoined, and let L_3 be the Boolean algebra with 3 atoms, as in Figure 6.1. Observe that L_3 is the distributive envelope of L_2, via the map g which sends x_2 to x_3, for $x \in \{a, b, c\}$. Let $f : L_1 \to L_2$ be the homomorphism defined by $f(x_1) = x_2$ for $x \in \{a, b\}$. The composition gf does not preserve (admissible) joins: $gf(a_1 \vee b_1) = gf(1) = 1$, but $gf(a_1) \vee gf(b_1) = a_3 \vee b_3 \neq 1$. Note that f, despite it being a homomorphism, does not send join-admissible sets to join-admissible sets: the image of $\{a_1, b_1\}$ is $\{a_2, b_2\}$, which is not join-admissible in L_2.
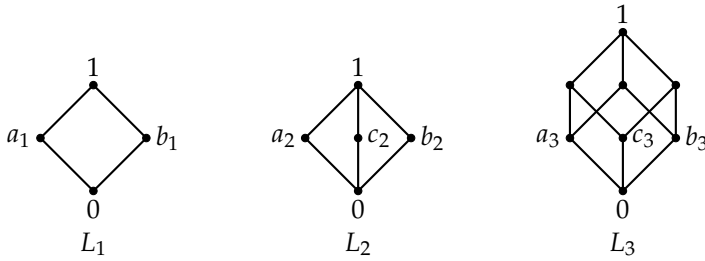
Figure 6.1: The lattices L_1, L_2 and L_3.

However, the following proposition shows that for *surjective* maps, the condition that f sends join-admissible sets to join-admissible sets can be omitted. It was already observed by Urquhart [139] that surjective maps are well-behaved for duality, and accordingly our duality in Section 6.2 will also include all surjective lattice homomorphisms.

Proposition 6.1.13. *Suppose $f : L_1 \to L_2$ is a surjective function which preserves finite meets and admissible joins. Then f sends join-admissible sets to join-admissible sets, and therefore f is a join-admissible morphism.*

Proof. Suppose that $M \subseteq L_1$ is a join-admissible set. To show that $f(M)$ is join-admissible, first let $a \in L_1$ be arbitrary. Note that it follows from the definition of join-admissibility that $\{a \wedge m \mid m \in M\}$ is also join-admissible in L_1. Using that f preserves meets and admissible joins, we get:

$$f(a) \wedge \bigvee_{m \in M} f(m) = f(a \wedge \bigvee M) = f\left(\bigvee_{m \in M} (a \wedge m) \right) = \bigvee_{m \in M} (f(a) \wedge f(m)).$$

Since f is surjective, any $b \in L_2$ is of the form $b = f(a)$ for some $a \in L_1$. Hence, $f(M)$ is join-admissible. $\square$

Note that, if L_1 and L_2 are distributive, then join-admissible morphisms from L_1 to L_2 are exactly bounded lattice homomorphisms. Hence, we have a full inclusion of categories $I^\wedge : \mathbf{DL} \hookrightarrow \mathbf{L_{a\vee}}$. The following is now a consequence of the universal property of the distributive envelope.

Corollary 6.1.14. *The functor $D^\wedge : \mathbf{L_{a\vee}} \to \mathbf{DL}$, which sends L to $D^\wedge(L)$ and a join-admissible morphism $f : L_1 \to L_2$ to the unique homomorphic extension of $\eta^\wedge_{L_2} \circ f : L_1 \to D^\wedge(L_2)$, is left adjoint to $I^\wedge : \mathbf{DL} \hookrightarrow \mathbf{L_{a\vee}}$ and $\eta^\wedge$ is the unit of the adjunction. Moreover, the counit $\epsilon^\wedge : D^\wedge \circ I \to 1_{\mathbf{DL}}$ is an isomorphism.*

Remark 6.1.15. We compare our results in this section so far to those of Bruns and Lasker [19]. The equivalence of (1) and (3) in Lemma 6.1.6 is

very similar to the statement of Lemma 3 in [19]. Our Corollary 6.1.9 is a finitary version of the characterisation in Corollary 2 of [19]. The fact that $D^\wedge$ is an adjoint to a full inclusion can also be seen as a finitary analogue of the result of [19] that their construction provides the injective hull of a meet-semilattice. Note that our construction of $D^\wedge(L)$ could also be applied to the situation where L is only a meet-semilattice, if we modify our definition of join-admissible sets to require that the relevant joins exist in L. The injective hull of L that was constructed in [19] can now be retrieved from our construction by taking the free directedly complete poset (dcpo) over the distributive lattice $D^\wedge(L)$. This is an instance of the general phenomenon that frame constructions may be seen as a finitary construction followed by a dcpo construction, cf. [87].

Remark 6.1.16. We outline the order-dual version of the construction given above for later reference. A finite subset $M \subseteq L$ is *meet-admissible* if, for all $a \in M$, we have $a \vee \bigwedge M = \bigwedge_{m \in M}(a \vee m)$. The *distributive $\vee$-envelope* is defined as in Definition 6.1.2, interchanging the words 'join' and 'meet' everywhere in the definition. An *a-filter* is an upset which is closed under admissible meets. The distributive $\vee$-envelope can be realized as the poset of finitely generated a-filters of L, ordered by reverse inclusion. The distributive $\vee$-envelope is also anti-isomorphic to the sublattice of $\mathcal{P}(M^\infty(L^\delta))$ that is generated by the sets

$$\breve{a} := \{y \in M^\infty(L^\delta) \mid a \leq y\},$$

by sending the a-filter generated by a finite set T to $\bigcup_{a \in T} \breve{a}$. Note that the order on a-filters has to be taken to be the reverse inclusion order, to ensure that the unit embedding $\eta_L^\vee$ of the adjunction will be order-preserving. On the other hand, the order in $\mathcal{P}(M^\infty(L^\delta))$ is the inclusion order, which explains why $D^\vee(L)$ is *anti*-isomorphic to a sublattice of $\mathcal{P}(M^\infty(L^\delta))$. We say that $f : L_1 \to L_2$ is a *meet-admissible morphism* if it preserves finite joins, admissible meets, and sends meet-admissible sets to meet-admissible sets. Then $D^\vee$ is a functor from the category $\mathbf{L_{a\wedge}}$ to $\mathbf{DL}$ which is left adjoint to the full and faithful functor $I^\vee : \mathbf{DL} \to \mathbf{L_{a\wedge}}$. We denote the unit of the adjunction by $\eta^\vee : 1_{\mathbf{L_{a\wedge}}} \to I^\vee D^\vee$. Finally, $D^\vee(L)$ is the (up to isomorphism) unique distributive meet-dense extension of L which preserves finite joins and admissible meets.

We end this section by examining additional structure which links the two distributive envelopes $D^\wedge(L)$ and $D^\vee(L)$, and enables us to retrieve L from the lattices $D^\wedge(L)$ and $D^\vee(L)$. Recall from Section 2.4 in Chapter 2 that a tuple (X, Y, R), where X and Y are sets and $R \subseteq X \times Y$ is a relation, is called a *polarity* and naturally induces a Galois connection[3] $u : \mathcal{P}(X) \leftrightarrows \mathcal{P}(Y) : l$.

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[3]We use the term *Galois connection* [14] for what is sometimes called a *contravariant adjunc-*

Let $u_L : \mathcal{P}(J^\infty(L^\delta)) \leftrightarrows \mathcal{P}(M^\infty(L^\delta)) : l_L$ be the Galois connection associated to the polarity $(J^\infty(L^\delta), M^\infty(L^\delta), \leq_{L^\delta})$, that is,

$$u_L(V) := \{y \in M^\infty(L^\delta) \mid \forall x \in V : x \leq_{L^\delta} y\} \qquad (V \subseteq J^\infty(L^\delta)),$$

$$l_L(W) := \{x \in J^\infty(L^\delta) \mid \forall y \in W : x \leq_{L^\delta} y\} \qquad (W \subseteq M^\infty(L^\delta)).$$

Note that if $V = \widehat{a}$ for some $a \in L$, then $u_L(V) = u_L(\widehat{a}) = \check{a}$. Recall from Corollary 6.1.10 that the distributive lattice $D^\wedge(L)$ is isomorphic to a sub-lattice of $\mathcal{P}(J^\infty(L^\delta))$, and, by Remark 6.1.16, $D^\vee(L)^{\mathrm{op}}$ is isomorphic to a sublattice of $\mathcal{P}(M^\infty(L^\delta))$.

Proposition 6.1.17. *For any lattice L, the maps u_L and l_L restrict to a Galois connection $u_L : D^\wedge(L) \leftrightarrows D^\vee(L)^{\mathrm{op}} : l_L$, whose lattice of Galois-closed elements is isomorphic to L.*

N.B. The restricted Galois connection in this proposition is between $D^\wedge(L)$ and the order dual of $D^\vee(L)$. Therefore, it is also a (covariant) adjunction between $D^\wedge(L)$ and $D^\vee(L)$.

Proof. The isomorphic copy of $D^\wedge(L)$ in $\mathcal{P}(J^\infty(L^\delta))$ consists of finite unions of sets of the form $\widehat{a}$. If $T \subseteq L$, then we have

$$u_L\left(\bigcup_{a \in T} \widehat{a}\right) = \bigcap_{a \in T} \check{a} = \check{t},$$

where $t := \bigvee T$. From this, it follows that $u_L(D^\wedge(L)) \subseteq D^\vee(L)$, and the analogous statement for l_L is proved similarly. The lattice of Galois-closed elements under this adjunction is both isomorphic to the image of u_L in $D^\vee(L)$ and the image of l_L in $D^\wedge(L)$. Both of these lattices are clearly isomorphic to L. □

In the presentation of $D^\wedge(L)$ and $D^\vee(L)$ as finitely generated a-ideals and a-filters, the maps u_L and l_L act as follows. Given an a-ideal I which is generated by a finite set $T \subseteq L$, $u_L(I)$ is the principal a-filter generated by $\bigvee T$. Conversely, given an a-filter F which is generated by a finite set $S \subseteq L$, $l_L(F)$ is the principal a-ideal generated by $\bigwedge S$.

In light of Proposition 6.1.17, we can combine $D^\wedge$ and $D^\vee$ to obtain a single functor, D, into a category of *adjoint pairs between distributive lattices*. On objects, this functor D sends a lattice L to the pair $u_L : D^\wedge(L) \leftrightarrows D^\vee(L) : l_L$ (see Proposition 6.1.17 above). For the morphisms in the domain category

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tion, i.e., a pair of order-preserving functions $u : P \leftrightarrows Q : l$ between posets satisfying $\mathrm{id}_Q \leq ul$ and $\mathrm{id}_P \leq lu$. We reserve the word adjunction for the covariant case. Both for a Galois connection and for an adjunction, we call Galois-closed those elements $p \in P$ such that $lu(p) = p$, and those $q \in Q$ such that $ul(q) = q$.

of D, we take the intersection of the set of morphisms in $\mathbf{L_{a\vee}}$ and the set of morphisms in $\mathbf{L_{a\wedge}}$. This intersection is defined directly in the following definition.

Definition 6.1.18. A function $f : L \to M$ between lattices is an *admissible homomorphism* if it is a lattice homomorphism which sends join-admissible subsets of L to join-admissible subsets of M and meet-admissible subsets of L to meet-admissible subsets of M. We denote by $\mathbf{L_a}$ the category of lattices with admissible homomorphisms.

Indeed, f is an admissible homomorphism if, and only if, it is a morphism both in $\mathbf{L_{a\vee}}$ and in $\mathbf{L_{a\wedge}}$. Any homomorphism whose codomain is a distributive lattice is admissible. Also, any surjective homomorphism between arbitrary lattices is admissible, by Proposition 6.1.13. This may be the underlying reason for the fact that both surjective homomorphisms and morphisms whose codomain is distributive have proven to be 'easier' cases in the existing literature on lattice duality (see, e.g., [139, 78]). Of course, not all lattice homomorphisms are admissible, cf. Example 6.1.12 above. In the next section, we will develop a topological duality for the category $\mathbf{L_a}$.

We end this section with a historical remark. The first construction of a canonical extension for lattices (although lacking an abstract characterization) was given in [77]. This construction depended on the fact that any lattice occurs as the Galois-closed sets of some Galois connection. In this section we have given a 'canonical' choice for this Galois connection. We will leave the precise statement of this last sentence to future work; also see the concluding section of this chapter.

6.2. Topological duality

In this section, we show how the above results can be applied to the topological representation theory of lattices. First, we will discuss how the existing topological dualities for lattices by Urquhart [139] and Hartung [78] relate to canonical extensions. We subsequently exploit this perspective on duality to obtain examples of lattices for which Hartung's dual space is not sober, or does not have a spectral soberification (Examples 6.2.1 and 6.2.2, respectively). The rest of the section will be devoted to obtaining an alternative topological duality for the category of lattices $\mathbf{L_a}$. In our duality, the spaces occurring in the dual category will be spectral.

As already remarked in [54, Remark 2.10], the canonical extension can be used to obtain the topological polarity in Hartung's duality for lattices [78]. We now briefly recall how this works. As is proved in [54, Lemma 3.4], the set $J^\infty(L^\delta)$ is in a natural bijection with the set of filters F which are

maximally disjoint from some ideal I, and the set $M^\infty(L^\delta)$ is in a natural bijection with the set of ideals which are maximally disjoint from some filter F. These are exactly the sets used by Hartung [78] in his topological representation for lattices. The topologies defined in [78] can be recovered from the embedding $L \hookrightarrow L^\delta$, as follows (cf. Figure 6.2).
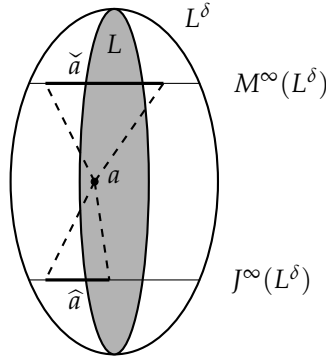
Figure 6.2: Topology from the embedding of a lattice L into its canonical extension L^δ.

For $a \in L$ we define $\widehat{a} := {\downarrow}a \cap J^\infty(L^\delta)$ and $\widecheck{a} := {\uparrow}a \cap M^\infty(L^\delta)$. Let τ^J be the topology on $J^\infty(L^\delta)$ given by taking $\{\widehat{a} : a \in L\}$ as a subbasis for the closed sets. Let τ^M be the topology on $M^\infty(L^\delta)$ given by taking $\{\widecheck{a} : a \in L\}$ as a subbasis for the closed sets. Finally, let R_L be the relation defined by $x\, R_L\, y$ if, and only if, $x \leq_{L^\delta} y$. This topological polarity $((J^\infty(L^\delta), \tau^J), (M^\infty(L^\delta), \tau^M), R_L)$ is now exactly (isomorphic to) Hartung's topological polarity $\mathbb{K}^\tau(L)$ in [78, Definition 2.1.6].

Before Hartung, Urquhart [139] had already defined the dual structure of a lattice to be a doubly ordered topological space $(Z, \tau, \leq_1, \leq_2)$ whose points are maximal filter-ideal pairs (F, I). We briefly outline how this structure can be obtained from the canonical extension. Let P be the subset of $J^\infty(L^\delta) \times M^\infty(L^\delta)$ consisting of pairs (x, y) such that $x \not\leq_{L^\delta} y$, i.e., P is the set-theoretic complement of the relation R_L in Hartung's polarity. Then P inherits the subspace topology from the product topology $\tau^J \times \tau^M$ on $J^\infty(L^\delta) \times M^\infty(L^\delta)$. We define an order $\preceq$ on P by $(x, y) \preceq (x', y')$ iff $x \geq_{L^\delta} x'$ and $y \leq_{L^\delta} y'$; in other words, $\preceq$ is the restriction of the product of the dual order and the usual order of L^δ. Urquhart's space (Z, τ) then corresponds to the subspace of $\preceq$-maximal points of P, and the orders $\leq_1$ and $\leq_2$ correspond to the projections of the order $\preceq$ onto the first and second coordinate, respectively.

In the following two examples, we prove that the spaces which occur in Hartung's duality may lack the properties of sobriety and arithmeticity (the fact that intersections of compact-opens are compact) that Stone dual spaces
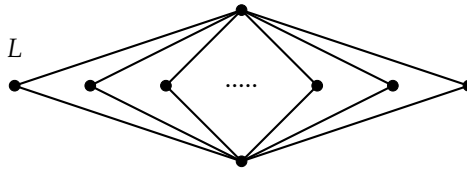
Figure 6.3: The lattice L, a countable antichain with top and bottom.

of distributive lattices always have.

Example 6.2.1 (A lattice for which Hartung's dual topology is not sober)**.** Let L be a countable antichain with top and bottom, as depicted in Figure 6.3. One may easily show that id : $L \to L$ is a canonical extension, so $L = L^\delta$. The set $J^\infty(L)$ is the countable antichain (as is the set $M^\infty(L)$). Note that the topology τ^J on $J^\infty(L)$ is the cofinite topology on a countable set, which is not sober: the entire space is itself a closed irreducible subset which is not the closure of a point. Also note that if one instead would define a topology on $J^\infty(L)$ by taking the sets $\hat{a}$, for $a \in L$, to be open, instead of closed, then one obtains the discrete topology on $J^\infty(L)$, which is not compact. $\square$

In light of the above example, one may wonder whether the *soberification* of the space $(J^\infty(L), \tau^J)$ might have better properties, and in particular whether it will be spectral. However, the following example shows that this is not always the case, since the frame of opens of the topological space $(J^\infty(L), \tau^J)$ in the following example fails to be arithmetic: intersections of compact-open sets are not necessarily compact.
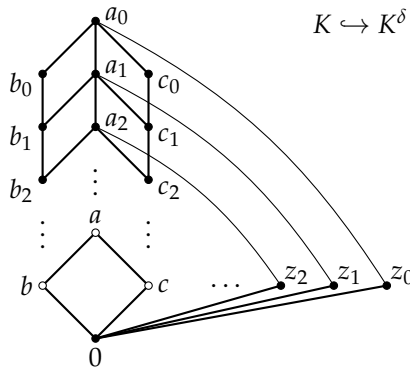
Figure 6.4: The lattice K, for which $(J^\infty(K^\delta), \tau^J)$ is not spectral.

Example 6.2.2 (A lattice for which Hartung's dual topology is not arithmetic)**.** Consider the lattice K depicted in Figure 6.4. In this figure, the elements of the original lattice K are drawn as filled dots, and the three additional elements a, b and c of the canonical extension K^δ are drawn as unfilled dots.

The set $J^\infty(K^\delta)$ is $\{b_i, c_i, z_i \mid i \geq 0\} \cup \{b, c\}$. Note that $(\widehat{b_0})^c$ and $(\widehat{c_0})^c$ are compact open sets in Hartung's topology τ^J. However, their intersection is not compact: $\{(\widehat{a_n})^c\}_{n=0}^\infty$ is an open cover of $(\widehat{b_0})^c \cap (\widehat{c_0})^c = \{z_i : i \geq 0\}$ with no finite subcover. $\qquad\square$

The above examples indicate that the spaces obtained in Hartung's duality can be badly behaved. In particular, they do not fit into the framework of the duality between sober spaces and spatial frames (cf. Section 1.2 in Chapter 1), and even their soberifications may fail to be the Stone duals of any distributive lattice.

In the remainder of this section, we combine the facts from Section 6.1 with the existing Stone-Priestley duality for distributive lattices to obtain a duality for a category of lattices with admissible homomorphisms (see Definition 6.1.18 below). Since the join-admissible morphisms are *exactly* the morphisms which can be lifted to homomorphisms between the $D^\wedge$-envelopes, these morphisms also correspond exactly to the lattice morphisms which have functional duals between the X-sets of the dual polarities; the same remark applies to meet-admissible morphisms and the Y-sets of the dual polarities. The duals of admissible morphisms will be pairs of functions; one function being the dual of the 'join-admissible part' of the morphism, the other being the dual of the 'meet-admissible part' of the morphism.

We will now first define an auxiliary category of 'doubly dense adjoint pairs between distributive lattices' (**daDL**) which has the following two features:

1. The category $\mathbf{L_a}$ can be embedded into **daDL** as a full subcategory (Proposition 6.2.4);

2. There is a natural Stone-type duality for **daDL** (Theorem 6.2.13).

We will then give a dual characterization of the 'special' objects in **daDL** which are in the image of the embedding of $\mathbf{L_a}$ from (1), calling these dual objects *tight* (cf. Definition 6.2.18). The restriction of the natural Stone-type duality (2) will then yield the main result of this section: a topological duality for lattices with admissible homomorphisms (Theorem 6.2.19).

Definition 6.2.3. We denote by **aDL** the category with:

- *objects:* tuples (D, E, f, g), where D and E are distributive lattices and $f : D \leftrightarrows E : g$ is a pair of maps such that f is lower adjoint to g;

- *morphisms:* an **aDL**-morphism from (D_1, E_1, f_1, g_1) to (D_2, E_2, f_2, g_2) is a pair of homomorphisms $h^\wedge : D_1 \to D_2$ and $h^\vee : E_1 \to E_2$ such that $h^\vee f_1 = f_2 h^\wedge$ and $h^\wedge g_1 = g_2 h^\vee$, i.e., both squares in the following diagram commute:

$$
\begin{array}{ccc}
D_1 & \xrightarrow{\quad g_1 \quad} & E_1 \\
& \overleftarrow{\quad f_1 \quad} & \\
h^\wedge \Big\downarrow & & \Big\downarrow h^\vee \\
& \xrightarrow{\quad g_2 \quad} & \\
D_2 & \overleftarrow{\quad f_2 \quad} & E_2
\end{array}
$$

We call an adjoint pair (D, E, f, g) *doubly dense* if both $g(E)$ is join-dense in D and $f(D)$ is meet-dense in E. We denote by **daDL** the full subcategory of **aDL** whose objects are doubly dense adjoint pairs.

Proposition 6.2.4. *The category* **L$_a$** *is equivalent to a full subcategory of* **daDL**.

Proof. Let $\mathcal{D} : \mathbf{L_a} \to \mathbf{daDL}$ be the functor defined by sending:

- a lattice L to $\mathcal{D}(L) := (D^\wedge(L), D^\vee(L), u, l)$,

- an admissible morphism $h : L_1 \to L_2$ to the pair $\mathcal{D}(h) := (D^\wedge(h), D^\vee(h))$.

We show that $\mathcal{D}$ is a well-defined full and faithful functor. For objects, note that $\mathcal{D}(L)$ is a doubly dense adjoint pair by Corollary 6.1.9 and Proposition 6.1.17 in the previous section. Let $h : L_1 \to L_2$ be an admissible morphism. We need to show that $\mathcal{D}(h)$ is a morphism of **daDL**, i.e., that $u_{L_2} \circ D^\wedge(h) = D^\vee(h) \circ u_{L_1}$ and $l_{L_2} \circ D^\vee(h) = D^\wedge(h) \circ l_{L_1}$. Since $D^\wedge(L_1)$ is join-generated by the image of L_1, and both $u_{L_2} \circ D^\wedge(h)$ and $D^\vee(h) \circ u_{L_1}$ are join-preserving, it suffices to note that the diagram commutes for elements in the image of L_1. This is done by the following diagram chase:

$$
u_{L_2} \circ D^\wedge(h) \circ \eta^\wedge_{L_1} = u_{L_2} \circ \eta^\wedge_{L_2} \circ h = \eta^\vee_{L_2} \circ h = D^\vee(h) \circ \eta^\vee_{L_1} = D^\vee(h) \circ u_{L_1} \circ \eta^\wedge_{L_1},
$$

where we have used that $\eta^\wedge$ is a natural transformation and that $u_L \circ \eta^\wedge_L = \eta^\vee_L$. The proof that $l_{L_2} \circ D^\vee(h) = D^\wedge(h) \circ l_{L_1}$ is similar.

It remains to show that the assignment $h \mapsto \mathcal{D}(h)$ is a bijection between $\mathbf{L_a}(L_1, L_2)$ and $\mathbf{daDL}(\mathcal{D}(L_1), \mathcal{D}(L_2))$. If $(h^\wedge, h^\vee) : \mathcal{D}(L_1) \to \mathcal{D}(L_2)$ is a **daDL**-morphism, then $h^\wedge$ maps lattice elements to lattice elements. That is, the function $h^\wedge \circ \eta^\wedge_{L_1} : L_1 \to \mathcal{D}(L_2)$ maps into $\operatorname{im}(\eta^\wedge_{L_2}) = \operatorname{im}(l_{L_2})$, since

$$
\begin{aligned}
h^\wedge \circ \eta^\wedge_{L_1} &= h^\wedge \circ l_{L_1} \circ u_{L_1} \circ \eta^\wedge_{L_1} \\
&= l_{L_2} \circ h^\vee \circ u_{L_1} \circ \eta^\wedge_{L_1}.
\end{aligned}
$$

We may therefore define $h : L_1 \to L_2$ to be the function $(\eta^\wedge_{L_2})^{-1} \circ h^\wedge \circ \eta^\wedge_{L_1}$. Note that this function is equal to $(\eta^\vee_{L_2})^{-1} \circ h^\vee \circ \eta^\vee_{L_1}$, since

$$
\begin{aligned}
(\eta^\vee_{L_2})^{-1} \circ h^\vee \circ \eta^\vee_{L_1} &= (\eta^\vee_{L_2})^{-1} \circ h^\vee \circ u_{L_1} \circ l_{L_1} \circ \eta^\vee_{L_1} \\
&= (\eta^\vee_{L_2})^{-1} \circ u_{L_2} \circ h^\wedge \circ \eta^\wedge_{L_1} \\
&= (\eta^\wedge_{L_2})^{-1} \circ h^\wedge \circ \eta^\wedge_{L_1},
\end{aligned}
$$

where we have used that, for any lattice L, $l_L \circ \eta_L^\vee = \eta_L^\wedge$ and $u_L \circ \eta_L^\wedge = \eta_L^\vee$. So, since $(\eta_{L_2}^\wedge)^{-1} \circ h^\wedge \circ \eta_{L_1}^\wedge = h = (\eta_{L_2}^\vee)^{-1} \circ h^\vee \circ \eta_{L_1}^\vee$, it is clear that h is a homomorphism, since the left-hand-side preserves $\wedge$ and the right-hand-side preserves $\vee$. It remains to show that h is admissible, i.e., that h sends join-admissible subsets to join-admissible subsets, and meet-admissible subsets to meet-admissible subsets. Note that, by the adjunction in Corollary 6.1.14, if a function $k : L \to D$ admits a homomorphic extension $\widehat{k} : D^\wedge(L) \to D$, then k is a join-admissible morphism, since it is equal to the composite $\widehat{k} \circ \eta_L^\wedge$. In particular, the morphism $\eta_{L_2}^\wedge \circ h$ is join-admissible, its homomorphic extension being $h^\wedge$. It follows from this that h sends join-admissible subsets to join-admissible subsets, since join-admissible subsets are the *only* subsets whose join is preserved by $\eta_{L_2}^\wedge$. The proof that h preserves meet-admissible subsets is similar. Now, since $h^\wedge \circ \eta_{L_1}^\wedge = \eta_{L_2}^\wedge \circ h$, we have that $h^\wedge = D^\wedge(h)$, since $D^\wedge(h)$ was defined as the unique homomorphic extension of $\eta_{L_2}^\wedge \circ h$, and similarly $h^\vee = D^\vee(h)$. We conclude that $(h^\wedge, h^\vee) = \mathcal{D}(h)$, so $h \mapsto \mathcal{D}(h)$ is surjective. It is clear that if $h \neq h'$, then $D^\wedge(h) \neq D^\wedge(h')$, so $\mathcal{D}(h) \neq \mathcal{D}(h')$. Hence, the assignment $h \mapsto \mathcal{D}(h)$ is bijective, as required. □

Example 6.2.5 (Not every object of **daDL** is the distributive envelope of a lattice). Let D be any distributive lattice and consider the daDL $(F_\vee(D, \wedge), F_\wedge(D, \vee), f, g)$, where $F_\vee(D, \wedge)$ is the free join-semilattice generated by the meet-semilattice reduct of D viewed as a distributive lattice, $F_\wedge(D, \vee)$ is defined order dually, and f and g both are determined by sending each generator to itself. Such a daDL is not of the form we are interested in since the $\wedge$- and $\vee$-envelopes of any distributive lattice both are equal to the distributive lattice itself, since all joins are admissible.

The above example shows that the category $\mathbf{L_a}$, that we will be most interested in, is a proper subcategory of **daDL**. We start by giving a description of the topological duals of the objects of **daDL**. To this end, let (D, E, f, g) be a doubly dense adjoint pair. If X and Y are the dual Priestley spaces of D and E respectively, then it is well-known that an adjunction (f, g) corresponds to a relation R satisfying certain properties. In our current setting of *doubly dense* adjoint pairs, it turns out that it suffices to consider the topological reducts of the Priestley spaces X and Y (i.e., forgetting the order) and the relation R between them. Both the Priestley orders of the spaces X and Y and the adjunction (f, g) can be uniquely reconstructed from the relation R, as we will prove shortly. The dual of a doubly dense adjoint pair will be a *totally separated compact polarity* (TSCP), which will be a polarity (X, Y, R), where X and Y are Boolean spaces and R is a relation from X to Y that satisfies certain properties (see Definition 6.2.6 for the precise definition).

We first fix some useful terminology for topological polarities, regarding the closure and interior operators induced by a polarity, its closed and open sets, and its associated quasi-orders. Let X and Y be sets and $R \subseteq X \times Y$. There is a closure operator $\overline{(\ )}$ on X given by

$$\overline{S} := \{x \in X \mid xR(\_) \subseteq SR(\_)\} \text{ for } S \subseteq X.$$

The subsets S of X satisfying $\overline{S} = S$ will be called *R-closed*. The R-closed subsets of X form a lattice in which the meet is intersection and join is the closure of the union. We also obtain an adjoint pair of maps:

$$\mathcal{P}(X) \overset{\Box}{\underset{\blacklozenge}{\rightleftarrows}} \mathcal{P}(Y)$$

given by

$$\blacklozenge S = SR(\_) = \{y \in Y \mid \exists x \in S \ xRy\}$$

and

$$\Box T = ((\_)R(T^c))^c = \{x \in X \mid \forall y \in Y(xRy \implies y \in T)\}.$$

The relation with the closure operator on X is that $\overline{S} = \Box \blacklozenge S$. Note also that on points of X this yields a quasi-order given by

$$x' \leq x \iff x'R(\_) \subseteq xR(\_).$$

Similarly, on Y we obtain an interior operator

$$T^\circ = \{y \in Y \mid \exists x \in X \left[xRy \text{ and } \forall y' \in Y(xRy' \implies y' \in T)\right]\} = \blacklozenge \Box T$$

and a quasi-order on Y given by

$$y \leq y' \iff (\_)Ry' \subseteq (\_)Ry.$$

The range of $\blacklozenge$ is equal to the range of the interior operator, and we call the sets in the range *R-open*. This collection of subsets of Y forms a lattice isomorphic to the lattice of R-closed subsets of X. In this incarnation, the join is given by union whereas the meet is given by interior of the intersection. Note that the R-closed subsets of X as well as the R-open subsets of Y all are downsets in the induced quasi-orders.

We are now ready to define the objects which will be dual to doubly dense adjoint pairs.

Definition 6.2.6. A *topological polarity* is a tuple (X, Y, R), where X and Y are topological spaces and R is a relation. A *compact polarity* is a topological polarity in which both X and Y are compact. A topological polarity is *totally separated* if it satisfies the following conditions:

1. *(R-separated)* The quasi-orders induced by R on X and Y are partial orders.

2. *(R-operational)* For each clopen downset U of X, the image $\blacklozenge U$ is clopen in Y; For each clopen downset V of Y, the image $\Box V$ is clopen in X;

3. *(Totally R-disconnected)* For each $x \in X$ and each $y \in Y$, if xRy does not hold, then there exist clopen sets $U \subseteq X$ and $V \subseteq Y$ with $\blacklozenge U = V$ and $\Box V = U$, such that $x \in U$ and $y \notin V$.

In what follows, we often abbreviate "totally separated compact polarity" to TSCP.

Remark 6.2.7. In the definition of totally separated topological polarities, the first property states that R separates the points of X as well as the points of Y. The second property states that R yields operations between the clopen downsets of X and of Y. Finally, the third property generalizes total order disconnectedness, well known from Priestley duality, hence the name total R-disconnectedness.

The following technical observation about total R-disconnectedness will be useful in what follows.

Lemma 6.2.8. *If a topological polarity* (X, Y, R) *is totally R-disconnected, then the following hold:*

- *If* $x' \not\leq x$ *then there exists* $U \subseteq X$ *clopen and R-closed such that* $x \in U$ *and* $x' \notin U$.

- *If* $y' \not\leq y$ *then there exists* $V \subseteq Y$ *clopen and R-open such that* $y \in V$ *and* $y' \notin V$.

Proof. Suppose that $x' \not\leq x$. By definition of $\leq$, there exists $y \in Y$ such that $x'Ry$ and $\neg(xRy)$. By total R-disconnectedness, there exist clopen U and V such that $\blacklozenge U = V$, $\Box V = U$, $x \in U$ and $y \notin V$. We now have $x' \notin U$, for otherwise we would get that $y \in \blacklozenge U = V$. Since $U = \Box V = \Box\blacklozenge U$, we get that U is R-closed, as required. The proof of the second property is dual. $\qquad\square$

Now, given a daDL (D, E, f, g), we call its *dual polarity* the tuple (X, Y, R), where X and Y are the topological reducts of the Priestley dual spaces of D and E, respectively (which are in particular compact), and R is the relation defined, for $x \in X$ and $y \in Y$, by

$$xRy \iff f(F_x) \subseteq F_y,$$

where F_x and F_y are the prime filters corresponding to the points x and y, respectively. Conversely, given a totally separated compact polarity (X, Y, R), we call its *dual adjoint pair* the tuple $(D, E, \blacklozenge, \Box)$, where D and E are the lattices of clopen downsets of X and Y in the induced orders, respectively, and $\blacklozenge$ and $\Box$ are the operations defined above (note that these operations are indeed well-defined by Remark 6.2.7). Note that if L is a distributive lattice, then its associated daDL is $(L, L, \mathrm{id}, \mathrm{id})$, which has dual polarity $(X, X, \leq)$, where $(X, \leq)$ is the usual Priestley dual space of L. Thus, the above definition generalizes the definition of the Priestley dual space.

The following three propositions constitute the object part of our duality for doubly dense adjoint pairs.

Proposition 6.2.9. *If (D, E, f, g) is a doubly dense adjoint pair, then its dual polarity (X, Y, R) is compact and totally separated.*

Proof. Let (D, E, f, g) be a doubly dense adjoint pair, and let L be the lattice which is isomorphic to both the image of g in D and to the image of f in E. The dual polarity (X, Y, R) is compact because the dual Priestley spaces of D and E are compact. For R-separation, suppose that $x \neq x'$ in X. We need to show that $xR(\_) \neq x'R(\_)$. Without loss of generality, pick $d \in D$ such that $d \in F_x$ and $d \notin F_{x'}$. Since L is join-dense in D and F_x is a prime filter, there exists $a \in L$ with $a \leq d$, such that $a \in F_x$. Note that $a \notin F_{x'}$ since $a \leq d$ and $d \notin F_{x'}$. It follows that $f(a) \notin f(F_{x'})$: if we would have $d' \in F_{x'}$ such that $f(a) = f(d')$, then we would get $d' \leq gf(d') = gf(a) = a$, contradicting that $a \notin F_{x'}$. By the prime filter theorem, there exists a prime filter $y \subseteq E$ such that $f(F_{x'}) \subseteq y$ and $f(a) \notin y$. Since we do have $f(a) \in f(F_x)$, it follows that $x'Ry$ and $\neg(xRy)$, so $x'R(\_) \neq xR(\_)$, as required. The proof that R induces a partial order on Y is similar. For R-operationality, it suffices to observe that, for any $d \in D$, we have $\blacklozenge \hat{d} = \hat{d}R(\_) = \widehat{f(d)}$ and, for any $e \in E$, we have $\Box \hat{e} = \widehat{g(e)}$. For total R-disconnectedness, suppose that xRy does not hold. This means that $f(F_x) \not\subseteq y$, so there is $d \in D$ such that $d \in x$ and $f(d) \notin y$. Since $d \leq gf(d)$, we get $gf(d) \in x$, so we may put $U := \widehat{gf(d)}$ and $V := \widehat{f(d)}$. $\qquad\Box$

Proposition 6.2.10. *If (X, Y, R) is a totally separated compact polarity, then its dual adjoint pair is doubly dense.*

Proof. From what was stated in the preliminaries above, it is clear that we get an adjoint pair between the lattices of clopen downsets. We need to show that it is doubly dense. To this end, let U be a clopen downset of X. We show that U is a finite union of clopen R-closed sets. First fix $x \in U$. For any $x' \notin U$, we have that $x' \not\leq x$. By Lemma 6.2.8, pick a clopen R-closed set $U_{x'}$ such that $x \in U_{x'}$ and $x' \notin U_{x'}$. Doing this for all $x' \notin U$, we obtain

a cover $\{U^c_{x'}\}_{x' \notin U}$ by clopen sets of the compact set U^c. Therefore, there exists a finite subcover $\{U^c_i\}^n_{i=1}$ of U^c. Let us write $V_x := \bigcap^n_{i=1} U_i$. We then get that $x \in V_x \subseteq U$, and V_x is clopen and R-closed, since each of the U_i is. Doing this for all $x \in U$, we get a cover $\{U_x\}_{x \in X}$ by clopen R-closed sets of the compact set U, which has a finite subcover. This shows that U is a finite union of clopen R-closed sets. The proof that clopen downsets of Y are finite intersections of clopen R-open sets is dual; we leave it to the reader.   □

Proposition 6.2.11. *Any totally separated compact polarity is isomorphic to its double dual. More precisely, if (X, Y, R) is a TSCP, let (X', Y', R') be the dual polarity of the dual adjoint pair of (X, Y, R). Then there are homeomorphisms $\varphi : X \to X'$, $\psi : Y \to Y'$ such that xRy if, and only if, $\varphi(x)R'\psi(y)$.*

Proof. Note that if (X, Y, R) is a TSCP, then X and Y with the induced orders are Priestley spaces: total-order-disconnectedness follows from Lemma 6.2.8 and the fact, noted above, that R-closed and R-open sets are downsets in the induced orders. Therefore, by Priestley duality we have homeomorphisms $\varphi : X \to X'$ and $\psi : Y \to Y'$, both given by sending points to their neighbourhood filters of clopen downsets. It remains to show that φ and ψ respect the relation R. By definition, we have $x'R'y'$ if, and only if, for any clopen downset U in $F_{x'}$, the set $\blacklozenge U$ is in $F_{y'}$. Suppose that xRy, and that $U \in F_{\varphi(x)}$. Then $x \in U$, so $y \in UR(\_)$, so $\blacklozenge U \in F_{\psi(y)}$. Conversely, suppose that xRy does not hold. By total R-disconnectedness, pick a clopen R-closed set U with $x \in U$ and $y \notin \blacklozenge U$. This set U is a clopen downset which witnesses that $\varphi(x)R'\psi(y)$ does not hold.   □

We can extend this object correspondence between daDL's and TSCP's to a dual equivalence of categories. The appropriate morphisms in the category of totally separated compact polarities are pairs of functions (s_X, s_Y), which are the Priestley duals of $(h^\wedge, h^\vee)$. The condition that morphisms in **daDL** make two squares commute (see Definition 6.2.3) dualizes to back-and-forth conditions on s_X and s_Y, as in the following definition.

Definition 6.2.12. A *morphism* in the category **TSCP** of totally separated compact polarities from (X_1, Y_1, R_1) to (X_2, Y_2, R_2) is a pair (s_X, s_Y) of continuous functions $s_X : X_1 \to X_2$ and $s_Y : Y_1 \to Y_2$, such that, for all $x \in X_1$, $x' \in X_2$, $y \in Y_1$, $y' \in Y_2$:

(forth) If $x R_1 y$, then $s_X(x) R_2 s_Y(y)$,

($\blacklozenge$-back) If $x' R_2 s_Y(y)$, then there exists $z \in X_1$ such that $z R_1 y$ and $s_X(z) \leq x'$,

($\square$-back) If $s_X(x) R_2 y'$, then there exists $w \in Y_1$ such that $x R_1 w$ and $y' \leq s_Y(w)$.

The conditions on these morphisms should look natural to readers who are familiar with back-and-forth conditions in modal logic. More detailed background on how these conditions arise naturally from the theory of canonical extensions can be found in [51, Section 5].

Theorem 6.2.13. *The category* **daDL** *is dually equivalent to the category* **TSCP**.

Proof. The hardest part of this theorem is the essential surjectivity of the functor which assigns to a daDL its dual polarity. We proved this in Proposition 6.2.11. One may then either check directly that the assignment which sends a **daDL**-morphism $(h^\wedge, h^\vee)$ to the pair (s_X, s_Y) of Priestley dual functions between the spaces in the dual polarities is a bijection between the respective sets of morphisms, or refer to [51, Section 5] for a more conceptual proof that uses canonical extensions. □

In particular, combining Theorem 6.2.13 with Proposition 6.2.4, the category $\mathbf{L_a}$ of lattices with admissible homomorphisms is dually equivalent to a full subcategory of **TSCP**. The task that now remains is to identify which totally separated compact polarities arise as duals of doubly dense adjoint pairs which are isomorphic to ones of the form $(D^\wedge(L), D^\vee(L), u_L, l_L)$ for some lattice L (not all doubly dense adjoint pairs are of this form; cf. Example 6.2.5).

Given any daDL (D, E, f, g), there is an associated lattice $L = \text{im}(g) \cong \text{im}(f)$ which embeds in D meet-preservingly and in E join-preservingly. We write $i : L \hookrightarrow D$ and $j : L \hookrightarrow E$ for the embeddings of L into D and E, respectively. These images generate D and E, respectively, because of the double denseness. However, the missing property is that i and j need not preserve admissible joins and meets, cf. Example 6.2.5. We will now give a dual description of this property. To do so, we will use the canonical extension of the adjunction $f : D \leftrightarrows E : g$ and of the embeddings i and j. For the definition of canonical extensions of maps we refer to Section 4.2 in Chapter 4, in particular equations (4.4)–(4.7). The definitions given there for distributive lattices still apply to the setting of lattices in general, cf. [54, Section 4]. All maps in our setting are either join- or meet-preserving, so that their σ- and π-extensions coincide ([54, Lemma 4.4]). We therefore denote the unique extension of a (join- or meet-preserving) map h by h^δ. Thus, we have maps $f^\delta : D^\delta \leftrightarrows E^\delta : g^\delta$, $i^\delta : L^\delta \to D^\delta$ and $j^\delta : L^\delta \to E^\delta$. For our dual characterization of double denseness, we will need the following basic fact, which is essentially the content of Remark 5.5 in [54].

Proposition 6.2.14. *Let* $f : D \leftrightarrows E : g$ *be an adjunction between distributive lattices and let* L *be the lattice of Galois-closed elements. Then the following hold:*

1. *$f^\delta : D^\delta \leftrightarrows E^\delta : g^\delta$ is an adjunction;*

2. *The image of g^δ forms a complete $\bigwedge$-subsemilattice of D^δ which is isomorphic, as a completion of L, to L^δ;*

3. *The image of f^δ forms a complete $\bigvee$-subsemilattice of E^δ which is isomorphic, as a completion of L, to L^δ.*

Proof. Item (1) is a special case of Proposition 4.2.1 in Chapter 4, cf. also [54, Prop. 6.6]. For (2), note first that the image of an upper adjoint between complete lattices always forms a complete $\bigwedge$-subsemilattice. To see that the image of g^δ is isomorphic to L^δ as a completion of L, it suffices by the uniqueness of canonical extensions (Theorem 1.2.3) to check that the natural embedding $L \hookrightarrow \text{im}(g^\delta)$ (given by the composition $L \hookrightarrow D \hookrightarrow D^\delta$) is compact and dense. Neither of these properties is hard to verify. The proof of item (3) is order-dual to (2). $\square$

Let M be a finite subset of the lattice L. Recall that by Lemma 6.1.5, M is join-admissible if, and only if, for each $x \in J^\infty(L^\delta)$, we have $x \leq \bigvee M$ implies $x \leq m$ for some $m \in M$. In order to translate this to a dual condition, it is useful to get a dual characterization of the elements of $J^\infty(L^\delta)$. In the following lemma, we will use the fact that the relation R can be alternatively defined using the lifted operation f: regarding X as $J^\infty(D^\wedge(L)^\delta)$ and Y as $J^\infty(D^\vee(L)^\delta)$, we have that $xRy \iff y \leq f^\delta(x)$.

Lemma 6.2.15. *Let (D, E, f, g) be a daDL, (X, Y, R) its dual polarity, $L \cong \text{im}(g) \cong \text{im}(f)$, with $i : L \hookrightarrow D$ and $j : L \hookrightarrow E$ the natural embeddings. Then the following hold:*

1. *For all $x \in J^\infty(D^\delta)$, there exists $x' \in F(L^\delta)$ such that $i^\delta(x') = x$.*

    For all $y \in M^\infty(E^\delta)$, there exists $y' \in I(L^\delta)$ such that $j^\delta(y') = y$.

2. *For all $x \in X = J^\infty(D^\delta)$, the following are equivalent:*

    (a) *$x \in i^\delta(J^\infty(L^\delta))$,*

    (b) *There exists $y \in Y$ such that xRy, and $\neg(x'Ry)$ for all $x' \in X$ with $x' < x$.*

3. *For all $y \in Y = J^\infty(E^\delta)$, the following are equivalent:*

    (a) *$\kappa(y) \in j^\delta(M^\infty(L^\delta))$,*

    (b) *There exists $x \in X$ such that xRy, and $\neg(xRy')$ for all $y' \in Y$ with $y' > y$.*

Proof. For item (1), let $x \in X = J^\infty(D^\delta)$. Then $x \in F(D^\delta)$, so x is equal $\bigwedge F$ for some filter F of D. For each $d \in F$, since $\text{im}(i) = \text{im}(g)$ is join-dense in D, we may pick a finite subset $S_d \subseteq L$ such that $d = \bigvee i(S_d)$. Let us write Φ

for the set of choice functions $F \to \bigcup_{d \in F} S_d$. Then, by distributivity of D^δ, we have

$$x = \bigwedge F = \bigwedge \{\bigvee i(S_d) \mid d \in F\} = \bigvee \{\bigwedge_{d \in F} i(\varphi(d)) \mid \varphi \in \Phi\}.$$

Since x is completely join-irreducible in D^δ, we get that $x = \bigwedge_{d \in F} i(\varphi(d))$ for some $\varphi \in \Phi$. Since i^δ is completely meet-preserving, $x = i^\delta(\bigwedge_{d \in F} \varphi(d))$. The proof of the second statement in (1) is order-dual. For (2), first note that the negation of (b) holds if, and only if, for all $y \in Y$ with $y \le f^\delta(x)$ there exists $x' < x$ such that $y \le f^\delta(x')$. Since Y is $\bigvee$-generating in E^δ, this condition is in turn equivalent to $f^\delta(x) \le \bigvee_{x' < x} f^\delta(x')$. Since f^δ is lower adjoint to g^δ by Proposition 6.2.14.1, we conclude that the negation of (b) is equivalent to:

$$x \le g^\delta f^\delta \left(\bigvee_{D^\delta} \{x' \in X \mid x' < x\} \right). \qquad (\star)$$

Note that the right-hand-side of $(\star)$ is $i^\delta(\bigvee_{L^\delta} \{v \in (i^\delta)^{-1}(X) \mid i^\delta(v) < x\})$, using item (1). Now suppose that (b) does not hold and that $x = i^\delta(u)$ for some $u \in L^\delta$. We must have $u \le \bigvee_{L^\delta} \{v \in (i^\delta)^{-1}(X) \mid i^\delta(v) < x\}$, using $(\star)$ and the fact that i^δ is injective (cf. [54, Lemma 4.9]). It follows that u is join-reducible. Conversely, suppose that $x \in X \setminus i^\delta(J^\infty(L^\delta))$. By item (1), pick $u \in F(L^\delta)$ such that $i^\delta(u) = x$. Since u is join-reducible, we have $u = \bigvee_{L^\delta} \{v \in J^\infty(L^\delta) \mid v < u\}$. Applying i^δ to both sides of this equality, we conclude that $(\star)$ holds, which implies the negation of (b). Item (3) is order-dual to item (2). □

In light of the above lemma, we make the following definitions.

Definition 6.2.16. Let (X, Y, R) be a TSCP. For $x \in X$, we say that x is *R-irreducible* if there exists $y \in Y$ such that xRy, and $\neg(x'Ry)$ for all $x' \in X$ with $x' < x$. Order dually, for $y \in Y$, we say that y is *R-irreducible* if there exists $x \in X$ such that xRy, and $\neg(xRy')$ for all $y' \in Y$ with $y' > y$. Let $U \subseteq X$ a clopen downset. We say that U is *R-regular* provided that, for each R-irreducible $x \in X$ with $xR(\_) \subseteq UR(\_)$, we have $x \in U$. Order dually, we say that a downset $V \subseteq Y$ is *R-coregular* provided that, for each R-irreducible $y \in Y$ with $(\_)Ry \subseteq (\_)RU$, we have $y \in U$.

To see the intuition behind this definition, suppose that (X, Y, R) is the TSCP dual to some lattice L. Note that, combining Lemma 6.2.15.2 with the characterization of join-admissibility in Lemma 6.1.5, we now get the following. Let $M \subseteq L$ be a finite set, and write $U_M := \bigcup_{m \in M} \widehat{m}$. The set M is join-admissible in L if, and only if, the set U_M is R-regular. A similar remark applies to meet-admissibility and R-coregularity.

Recall that a clopen downset $U \subseteq X$ is R-closed provided that, for each $x \in X$, we have $xR(\_) \subseteq UR(\_)$ implies $x \in U$. Thus it is clear that every R-closed clopen downset in X is R-regular. Preserving admissible joins exactly corresponds to the reverse implication: as soon as U is R-regular it must also be R-closed. To sum up:

Proposition 6.2.17. *Let (D, E, f, g) be a daDL, and let (X, Y, R) be its dual polarity. Then the following are equivalent:*

1. *There exists a lattice L such that $(D, E, f, g) \cong (D^\wedge(L), D^\vee(L), u_L, l_L)$;*

2. *The embedding $\mathrm{im}(g) \hookrightarrow D$ preserves admissible joins and the embedding $\mathrm{im}(f) \hookrightarrow E$ preserves admissible meets.*

3. *In (X, Y, R), all R-regular clopen downsets in X are R-closed, and all R-coregular clopen downsets in Y are R-open.*

Proof. The equivalence of (1) and (2) holds by Corollary 6.1.9, its order dual, and the definition of the Galois connection (u_L, l_L). We now prove that (2) and (3) are equivalent. Throughout the proof, we write L for the lattice $\mathrm{im}(g)$, in which meets are given as in D and $\bigvee_L S = gf(\bigvee_D S)$, for any $S \subseteq L$. In this proof, we regard L as a sublattice of D, suppressing the notation i for the embedding $L \hookrightarrow D$. First suppose (2) holds, and let U be an R-regular clopen downset in X. Since the image of g is $\bigvee$-dense in D, there exists $M \subseteq \mathrm{im}(g)$ such that $U = \bigcup_{m \in M} \widehat{m}$. We show that M is join-admissible in the lattice L, using Lemma 6.1.5. If $x \in J^\infty(L^\delta)$ and $x \leq \bigvee_L M = gf(\bigvee_D M)$, then we have $f^\delta(x) \leq f(\bigvee_D M)$ by adjunction. By definition of R and the fact that f is completely join-preserving, we get that $xR(\_) \subseteq UR(\_)$. Since U is R-regular and $x \in J^\infty(L^\delta)$, we get that $x \in U$, so $x \leq m$ for some $m \in M$. Hence, M is join-admissible, and (2) implies that $\bigvee_L M = gf(\bigvee_D M) = \bigvee_D M$. That is, $\overline{U} = U$, so U is R-closed. The proof that R-coregular clopen downsets in Y are R-open is dual. Now suppose that (3) holds, and let $M \subseteq L$ be a join-admissible subset. Write U for the clopen downset $\bigcup_{m \in M} \widehat{m}$ in X. We show that U is R-regular. Let $x \in X$ be R-irreducible and suppose that $xR(\_) \subseteq UR(\_)$. Then $x \in J^\infty(L^\delta)$ and $f^\delta(x) \leq f(\bigvee_D M)$, so $x \leq gf(\bigvee_D M) = \bigvee_L M$. So, since M is join-admissible, there exists $m \in M$ such that $x \leq m$. In particular, we have $x \in U$, as required. By the assumption (3), we conclude that U is R-closed, i.e., $\overline{U} = U$, so that $\bigvee_L M = gf(\bigvee_D M) = \bigvee_D M$. The proof that $\mathrm{im}(f) \hookrightarrow E$ preserves admissible meets is dual. $\square$

In light of this proposition, we can now define a subcategory of TSCP's which will be dual to the category of lattices with admissible homomorphisms.

Definition 6.2.18. Let (X, Y, R) be a TSCP. We say that (X, Y, R) is *tight* if all R-regular clopen downsets in X are R-closed, and all R-coregular clopen downsets in Y are R-open. We denote by **tTSCP** the full subcategory of **TSCP** whose objects are the tight TSCP's.

We now obtain our topological duality theorem for lattices with admissible homomorphisms.

Theorem 6.2.19. *The category $\mathbf{L_a}$ of lattices with admissible homomorphisms is dually equivalent to the category $\mathbf{tTSCP}$ of tight totally separated compact polarities.*

Proof. By Proposition 6.2.4, we have that $\mathbf{L_a}$ is equivalent to a full subcategory of **daDL**. By Theorem 6.2.13, the category **daDL** is dually equivalent to **TSCP**. Proposition 6.2.17 shows that the image of $\mathbf{L_a}$ in **daDL** under this dual equivalence is **tTSCP**. □

The above theorem is not as general as possible: although we have only developed a duality for $\mathbf{L_a}$ here, it should be possible to generalize this duality to the categories $\mathbf{L_{a\vee}}$ and $\mathbf{L_{a\wedge}}$. To do so, one would need to generalize the category **tTSCP** to one where the morphisms are single functions instead of pairs of functions. We leave this to future work.

In this section, in light of Examples 6.2.1 and 6.2.2, we set out to obtain a topological duality for lattices in which the spaces are nicer than those occurring in Hartung's duality. Although the spaces obtained in our duality are as nice as can be (they are compact, Hausdorff and totally disconnected), this comes at the price of a rather complicated characterization. Therefore, we are inclined to draw as a negative conclusion that topology may not be the most opportune language to discuss 'duality' for lattices (unless the definition of a tTSCP can be simplified). Fortunately, the perspective of canonical extensions provides an alternative to topology. We have explained above how canonical extensions can be viewed as a point-free version of Hartung's duality, and we have used them to reason about the topological dual spaces introduced in this chapter. In the last section of this chapter, we will propose quasi-uniform spaces as a "spatial" alternative to topology in the context of set-theoretic representations of lattices.

6.3. Quasi-uniform spaces associated with a lattice

In this section we will show that the distributive envelopes of a lattice, which were defined by a universal property in Section 6.1, are also natural from a generalized topological perspective. The appropriate framework is that of quasi-uniform spaces, which generalize both quasi-orders

and topologies (see [47], in particular Chapter 3, for background on the theory of quasi-uniform spaces used in this section). In this section we will associate two Pervin quasi-uniform spaces to a lattice L, and then show in Theorem 6.3.3 that the completions of these quasi-uniform spaces coincide with the dual spaces of the distributive envelopes of L. Thus, quasi-uniform spaces give a precise spatial meaning to the distributive envelopes of L. Note that Pervin spaces, uniform completions and compactifications were used by Erné and Palko [41, 43] to obtain order-theoretic ideal completions. In this section, we will use the main result from [53, Section 1], which relates dual spaces to completions of uniform spaces, cf. Theorem 6.3.2 below. Given a set X, we denote, for each subset $A \subseteq X$, by U_A the subset

$$(A^c \times X) \cup (X \times A) = \{(x,y) \mid x \in A \implies y \in A\}$$

of $X \times X$. Given a topology τ on X, the filter $\mathcal{U}_\tau$ in the power set of $X \times X$ generated by the sets U_A for $A \in \tau$ is a totally bounded transitive quasi-uniformity on X [47, Proposition 2.1]. The quasi-uniform spaces $(X, \mathcal{U}_\tau)$ were first introduced by Pervin [125] and are now known in the literature as Pervin spaces. Generalizing this idea (also see [35]), given any subcollection $C \subseteq \mathcal{P}(X)$, we define $(X, \mathcal{U}_C)$ to be the quasi-uniform space whose quasi-uniformity is the filter generated by the entourages U_A for $A \in C$. Here we will call this larger class of quasi-uniform spaces *Pervin spaces*.

The first crucial point is that, for any collection $C \subseteq \mathcal{P}(X)$, the bounded distributive sublattice $D(C)$ of $\mathcal{P}(X)$ generated by C may be recovered from $(X, \mathcal{U}_C)$, even though this cannot be done in general from the associated topology. The *blocks* of a space $(X, \mathcal{U})$ are the subsets $A \subseteq X$ such that U_A is an entourage of the space, or equivalently, those for which the characteristic function $\chi_A : X \to 2$ is uniformly continuous with respect to the Sierpiński quasi-uniformity on 2, which is the one containing just 2^2 and $\{(0,0), (1,1), (1,0)\}$. The following fact is well-known, but we give a proof since it does not seem to be readily available in the literature.

Theorem 6.3.1. *Let X be a set and $C \subseteq \mathcal{P}(X)$ a collection of subsets. The set of blocks of the quasi-uniform space $(X, \mathcal{U}_C)$ is the sublattice $D(C)$ of $\mathcal{P}(X)$ generated by C.*

Proof. The blocks of any quasi-uniform space form a lattice, since $U_A \cap U_B$ is contained in both $U_{A \cap B}$ and $U_{A \cup B}$, for any $A, B \subseteq X$. If A is a block of $\mathcal{U}_C$, then by definition U_A contains a set of the form $\bigcap_{B \in F} U_B$, where $F \subseteq C$ is finite. From this, it follows that $A = \bigcup\{\bigcap\{B \mid x \in B, B \in F\} \mid x \in A\}$ (cf. [35, Lemma 2]). $\qquad\square$

Further, it is not hard to see that if $D \subseteq \mathcal{P}(Y)$ and $E \subseteq \mathcal{P}(X)$ are bounded sublattices of the respective power sets, then a map $f : (X, \mathcal{U}_E) \to (Y, \mathcal{U}_D)$ is

uniformly continuous if and only if f^{-1} restricts to a lattice homomorphism from D to E. Thus, the category of sublattices of power sets with morphisms that are commuting diagrams

$$
\begin{array}{ccc}
D & \xrightarrow{h} & E \\
\Big\uparrow & & \Big\downarrow \\
\mathcal{P}(Y) & \xrightarrow{\varphi} & \mathcal{P}(X),
\end{array}
$$

where φ is a complete lattice homomorphism, is dually isomorphic to the category of Pervin spaces with uniformly continuous maps.

To be able to state the main result from [53] that we want to apply here, we need to recall the definition of *bicompletion* of a quasi-uniform space. For more details see [47, Chapter 3]. Bicompleteness generalizes the notion of completeness for uniform spaces, which is well-understood (see, e.g., [17, Chapter II.3]): a uniform space $(X, \mathcal{U})$ is *complete* if every Cauchy filter converges. Now let $(X, \mathcal{U})$ be a quasi-uniform space. A quasi-uniform space $(X, \mathcal{U})$ is called *bicomplete* if and only if its symmetrization $(X, \mathcal{U}^s)$ is a complete uniform space. Here, recall that the *symmetrization*, $\mathcal{U}^s$, of the quasi-uniformity $\mathcal{U}$ is defined as the filter of $\mathcal{P}(X \times X)$ generated by the union of $\mathcal{U}$ and $\mathcal{U}^{-1}$. It has been shown by Fletcher and Lindgren [47, Chapter 3.3] that the full subcategory of bicomplete quasi-uniform spaces forms a reflective subcategory of the category of quasi-uniform spaces with uniformly continuous maps. Thus, for each quasi-uniform space $(X, \mathcal{U})$, there is a bicomplete quasi-uniform space $(\widetilde{X}, \widetilde{\mathcal{U}})$ and a uniformly continuous map $\eta_X : (X, \mathcal{U}) \to (\widetilde{X}, \widetilde{\mathcal{U}})$ with an appropriate universal property.

Theorem 6.3.2 ([53], Theorem 1.6). *Let D be a bounded distributive lattice, and let $e : D \hookrightarrow \mathcal{P}(X)$ be any bounded lattice embedding of D in a power set lattice. Denote by $\mathcal{D}$ the image of the embedding e. Let $\widetilde{X}$ be the bicompletion of the Pervin space $(X, \mathcal{U}_\mathcal{D})$. Then $\widetilde{X}$ with the induced topology is the Stone dual space of D.*

Alternatively, one can think of the quasi-uniform space $(\widetilde{X}, \widetilde{\mathcal{U}}_\mathcal{D})$ as an ordered uniform space, as follows. Equip the uniform space $(\widetilde{X}, \widetilde{\mathcal{U}_\mathcal{D}^s})$ with the order $\leq$ defined by $\bigcap_{a \in \mathcal{D}} U_{\hat{a}}$. Then $(\widetilde{X}, \widetilde{\mathcal{U}_\mathcal{D}^s}, \leq)$ is a uniform version of the *Priestley dual space* of D.

We now apply Theorem 6.3.2 to the setting of this chapter. Let L be a bounded lattice with dual polarity (X_L, Y_L, R_L). Then L induces quasi-uniform space structures $(X_L, \mathcal{U}_{\hat{L}})$ and $(Y_L, \mathcal{U}_{\check{L}})$ on X_L and Y_L, respectively. Here $\mathcal{U}_{\hat{L}}$ is the Pervin quasi-uniformity generated by the image $\hat{L} = \{\hat{a} \mid a \in L\}$ and $\mathcal{U}_{\check{L}}$ is the Pervin quasi-uniformity generated by the image $\check{L} = \{\check{a} \mid a \in L\}$. By Theorem 6.3.2, the bicompletions of these Pervin spaces are spectral

spaces and the corresponding bounded distributive lattices are the sublat-
tices of $\mathcal{P}(X_L)$ and $\mathcal{P}(Y_L)$ generated by $\widehat{L}$ and $\widecheck{L}$, respectively. The follow-
ing theorem now follows by combining Corollary 6.1.10, Theorem 6.1.4 and
Theorem 6.3.2.

Theorem 6.3.3. *Let L be a lattice. The bicompletion of the associated quasi-
uniform Pervin space, $(X_L, \mathcal{U}_{\widehat{L}})$, is the dual space of the distributive $\wedge$-envelope,
$D^{\wedge}(L)$, of L. Order dually, the bicompletion of the quasi-uniform Pervin space
$(Y_L, \mathcal{U}_{\widecheck{L}})$ is the dual space of the distributive $\vee$-envelope, $D^{\vee}(L)$, of L.*

Example 6.3.4. For any finite lattice L, the distributive envelope $D^{\wedge}(L)$ is
the lattice of downsets of the poset $J(L)$, with the order inherited from L.
Thus, in the finite case, the quasi-uniform space X_L is already bicomplete,
and hence equal to its own bicompletion. The same of course holds for
$D^{\vee}(L)$ and Y_L. In the finite case, X_L and Y_L are just the spaces occurring
in Hartung's duality. For the lattice L discussed in Example 6.2.1 above,
the distributive envelope $D^{\wedge}(L)$ is (isomorphic to) the lattice consisting of
all finite subsets of the countable antichain, and a top element. Thus, in
the bicompletion of X_L, we find one new point, corresponding to the prime
filter consisting of only the top element. For the lattice K discussed in Ex-
ample 6.2.2, the distributive envelope $D^{\wedge}(K)$ is a much bigger lattice than
K, and the bicompletion of X_L will contain many new points. In particular,
the bicompletion will not just be the soberification of X_L.

Concluding remarks

In this chapter, we developed the theory of distributive envelopes and used
it to obtain a topological duality for lattices. We see our methodology as
an example of the phenomenon that canonical extensions and duality may
help to study lattice-based algebras, even when they do not lie in finitely
generated varieties. As a case in point, our proof of the existence of dis-
tributive envelopes in Section 6.1 made use of canonical extensions of lat-
tices as a key tool. Moreover, the work in that section enabled us to identify
the join-admissible morphisms between lattices. In Section 6.2, we saw that
join- and meet-admissible morphisms are exactly the ones which have func-
tional duals on the X- and Y-components of the dual polarities. We believe
that canonical extensions may be used in a similar way for other varieties
of algebras based on lattices, such as residuated lattices, to mention just one
example.

In Section 6.3, we provided an alternative view of set-representation of lat-
tices, which replaces topology by quasi-uniformity and completion. Theo-
rem 6.3.3 opens the way for obtaining an alternative duality for lattices, in

which quasi-uniform spaces take the place of topological spaces. To do so, an interesting first step would be to represent the adjunction $D^{\wedge}(L) \leftrightarrows D^{\vee}(L)$ as additional structure on the pair of quasi-uniform spaces. We leave the development of these ideas to further research.

Let us mention one more possible direction for further work. For distributive lattices, the canonical extension functor is left adjoint to the inclusion functor of perfect distributive lattices into distributive lattices. However, this is known to be true for lattice-based algebras only in case all basic operations are both Scott and dually Scott continuous (see [32, Proposition C.9, p. 196] for a proof in the distributive setting). It follows from results of Goldblatt [71] that the canonical extension functor for modal algebras (i.e., Boolean algebras equipped with a modal operator) can be viewed as a left adjoint. However, the codomain category that is involved here is not immediately obvious: it is not the category of 'perfect modal algebras' in the usual sense. We conjecture that the distributive envelope constructions developed in Section 6.1 of this chapter may be used to define a category in which the canonical extension for lattices is a left adjoint. We also leave the actual development of this line of thought to future research.

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Notation

Index

Abstract

Algebra and logic

Logic, in the broadest sense of the word, is the study of reasoning. Algebra is famous for having a somewhat peculiar, often feared syntax, which was invented to abstract the language of everyday calculations and led to expressions such as $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. The core idea in the algebraic study of logic is that one can calculate with sentences and other linguistic expressions *just as if they were numbers*. This revolutionary insight is due to the 19th century English logician George Boole [15, 21].

Boole's algebraization of logic made it possible to investigate, with mathematical methods, the basic rules and assumptions that govern reasoning. From the early 20th century onwards, many other logical systems were introduced that departed from Boole's original system. Among these were the so-called *intuitionistic* logic, stemming from the work of the Dutch mathematician and philosopher L. E. J. Brouwer, and several *many-valued* logics were proposed, among others, by Kurt Gödel and Jan Łukasiewicz. The algebraic structure that arose from several of these "modern" kinds of logic plays a role in several chapters in this thesis.[1]

Duality

One of the main themes in this thesis is *duality*. Duality is a mathematical framework which studies the fundamental connection between form (*syntax*) and meaning (*semantics*) in logic. The first important insight of duality theory, due to M. H. Stone in his foundational work [137], was that Boole's algebraic system of logic could be alternatively represented with *topology*: the mathematical theory of the space around us. Stone's duality thus exhibited a deep connection between a formulaic, syntactic kind of reasoning on the one hand, and a more visual, spatial kind of reasoning on the other. Crucially, Stone noticed that the move between the two kinds of reasoning involves a *reversal* in the direction of transformations, in a way analogous to the classical, 19th-century 'Galois theory' of field extensions and their Galois groups.

The beautiful theory of duality which emerged from Stone's work has since been generalized and fine-tuned in several different directions, so that it can also accommodate the modern strands of logic that we already men-

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[1]To give just one example: MV-algebras (short for "many-valued algebras"), which are the object of study in Chapter 4 of this thesis, were introduced to study the version of many-valued logic introduced by Łukasiewicz.

tioned above. Two of these directions are of particular importance to much of the work in this thesis. First, H. A. Priestley's work [127], which showed that also *order*, not only topology, often plays an important role in duality. Second, the theory of *canonical extensions*, which casts duality theory itself in algebraic form, initiated by Jónsson and Tarski in the 1950's [84] and continued by Jónsson, Gehrke [56], and several other authors since the 1990's. In Chapter 1 of this thesis, we discuss Stone's and Priestley's dualities and the theory of canonical extensions in more detail. We then generalize these two techniques in Chapter 2 to the domain of *stably compact spaces*.

Sheaves

The French mathematician A. Grothendieck and his "Séminaire de Géométrie Algébrique du Bois Marie" [75] pioneered algebraic geometry in the 1960's, using the then recently developed theory of sheaves[2] as an essential tool. A *sheaf* is used in algebraic geometry to gain understanding of a complex algebraic structure by arranging it into several smaller, usually simpler, pieces. This way, one can often obtain local information about the algebraic structure which was hard to grasp by only looking at the structure as a whole.

Sheaves and duality have something in common: both methods use spatial insights to study structures of an algebraic nature. Indeed, a central result in this thesis (Theorem 3.3.7 in Chapter 3) makes this connection between Grothendieck's sheaves and Stone's duality mathematically precise. This theorem is subsequently applied in Chapter 4 to the special case of MV-algebras, already mentioned in note 1 above. A different connection between sheaves and duality is made in Chapter 5, where we develop Priestley duality for *skew lattices*. In the last chapter of this thesis, Chapter 6, we develop a *distributive envelope* for *lattices*, and apply it to the study of duality for lattices.

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[2]In everyday language, the word "sheaf" is sometimes used in expressions such as "sheaf of light", or "sheaf of corn", to mean approximately the same thing as "bundle".

Samenvatting

Algebra en logica

Logica, in de breedste zin van het woord, bestudeert het redeneren. Algebra staat bekend om haar enigszins vreemde, vaak gevreesde syntax, die werd uitgevonden om de taal van alledaagse berekeningen te abstraheren, en leidde tot uitdrukkingen zoals $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. Het kern-idee in de algebraïsche bestudering van logica is dat men met zinnen en andere taalkundige uitdrukkingen kan rekenen *alsof het getallen zijn*. Dit revolutionaire inzicht is afkomstig van de 19^e-eeuwse Engelse logicus George Boole [15, 21]. Boole's algebraïsering van logica maakte het mogelijk om met wiskundige methoden de basisregels en -aannames te bestuderen die ten grondslag liggen aan het redeneren. Vanaf de vroege 20^e eeuw werden er veel andere logische systemen geïntroduceerd die afweken van Boole's oorspronkelijke systeem. Hieronder bevonden zich de zogenaamde *intuïtionistische* logica, afkomstig uit het werk van de Nederlandse wiskundige en filosoof L. E. J. Brouwer, en verschillende *meerwaardige* logica's die werden voorgesteld door onder andere Kurt Gödel en Jan Łukasiewicz. De algebraïsche structuur die naar boven kwam uit verschillende van deze "moderne" soorten logica speelt een rol in verschillende hoofdstukken in dit proefschrift.[1]

Dualiteit

Een van de voornaamste thema's in dit proefschrift is *dualiteit*. Dualiteit is een wiskundig raamwerk dat de fundamentele verbintenis tussen vorm (*syntax*) en betekenis (*semantiek*) in de logica bestudeert. Het eerste belangrijke inzicht in dualiteitstheorie werd beschreven door M. H. Stone in zijn baanbrekende artikel [137]. Stone liet zien dat Boole's algebraïsche systeem voor logica ook kon worden gerepresenteerd met *topologie*: de wiskundige theorie van de ruimte om ons heen. De dualiteit van Stone bracht op deze manier een diepe verbintenis naar voren tussen aan de ene kant een formulaïsche, syntactische vorm van redeneren, en aan de andere kant een meer visuele, ruimtelijke variant. Verder had Stone het cruciale inzicht dat het heen en weer gaan tussen deze twee soorten van redeneren een *omkering* in de richting van transformaties met zich meebrengt, op een wijze analoog aan de klassieke, 19^e-eeuwse 'Galois-theorie' van lichaamsuitbreidingen en hun Galois-groepen.

De prachtige theorie van dualiteit die voortkwam uit het werk van Stone

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[1]Om hier een voorbeeld van te geven: MV-algebra's (een afkorting voor "many-valued algebras"), die in Hoofdstuk 4 van dit proefschrift aan bod komen, werden geïntroduceerd om Łukasiewicz' meerwaardige logica te bestuderen.

is sindsdien veralgemeniseerd en geraffineerd in verschillende richtingen, waardoor de theorie nu ook toepasbaar is op de hierboven al genoemde modernere vormen van logica. Twee van deze richtingen zijn van bijzonder belang voor een groot deel van dit proefschrift. Ten eerste het werk van H. A. Priestley [127], waarin zij aantoonde dat niet alleen topologie, maar ook *ordening*, vaak een belangrijke rol in dualiteit speelt. Ten tweede is de theorie van canonieke extensies, die de dualiteitstheorie zelf in een algebraïsche vorm giet, van belang. De theorie van canonieke extensies werd geïnitieerd door Jónsson en Tarski in de jaren '50 van de vorige eeuw [84], en voortgezet door Jónsson, Gehrke [56], en vele andere auteurs sinds de jaren '90. In Hoofdstuk 1 van dit proefschrift worden Stone's en Priestley's dualiteiten en de theorie van canonieke extensies in meer detail besproken. Vervolgens generaliseren we deze twee technieken in hoofdstuk 2 tot het domein van de *stably compact spaces* (stabiel compacte ruimtes).

Schoven

De Franse wiskundige A. Grothendieck vervulde in de jaren '60 van de vorige eeuw een pioniersrol in de algebraïsche meetkunde met zijn "Séminaire de Géométrie Algébrique du Bois Marie" [75], waarin de toen recent ontwikkelde schoventheorie een essentieel instrument was. Een *schoof* [2] wordt in de algebraïsche meetkunde gebruikt om een complexe algebraïsche structuur beter te begrijpen door haar te rangschikken met behulp van verschillende kleinere, vaak eenvoudigere, stukken. Op deze manier kan men lokale informatie over de algebraïsche structuur ontdekken die niet zichtbaar is als er louter naar de globale structuur wordt gekeken.

Schoven en dualiteit hebben iets gemeen: beide methoden gebruiken ruimtelijke inzichten om structuren van algebraïsche aard te bestuderen. Een centraal resultaat in dit proefschrift (Stelling 3.3.7 in Hoofdstuk 3) laat zien dat dit verband tussen Grothendieck's schoven en Stone's dualiteit inderdaad wiskundig hard gemaakt kan worden. Vervolgens wordt deze stelling in Hoofdstuk 4 toegepast op het bijzondere geval van MV-algebra's, al genoemd in noot 1 hierboven. Een ander verband tussen schoven en dualiteit wordt gelegd in Hoofdstuk 5, waar een Priestley dualiteit voor *skew lattices* (scheve tralies) wordt ontwikkeld. In het laatste hoofdstuk van dit proefschrift, Hoofdstuk 6, ontwikkelen we een *distributive envelope* (distributieve omhulling) voor *lattices* (tralies) en passen we deze toe om dualiteit voor tralies te bestuderen.

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[2]In alledaags taalgebruik wordt het woord "schoof" soms gebruikt in uitdrukkingen zoals "een schoof licht", of "een schoof koren", en betekent dan ongeveer hetzelfde als "bundel".

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Sam van Gool
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About the author

Sam van Gool was born in Amsterdam in 1987. After attending the Vossius Gymnasium high school in Amsterdam, he studied mathematics, classics, and logic, graduating *with distinction* in mathematics at the University of Cambridge in 2008 and *cum laude* in logic at the University of Amsterdam in 2009. Between 2010 and 2013 he was a PhD student in the NWO-funded project "Mathematically and Computationally Relevant Dualities" at the Radboud University Nijmegen, advised by Mai Gehrke. Throughout this period, he made frequent visits to the LIAFA laboratory of the Université Paris-Diderot in Paris.

Aside from his interest in mathematics and logic, Sam has a fascination for the inner workings of languages and music. He wrote a short opera, "Orpheus in Arezzo", for women's choir, two soloists and wind ensemble, which had its world premiere in the Rijksmuseum voor Oudheiden in Leiden in 2012.