Models of computation and finite automata

(MPRI course 2.16, 2021-2022)

Part II: Automata, Monoids and Logic - Problem Sheet 3

Finite $\mathcal{J}$-trivial monoids

Exercise 1. Recall that $a_1 \dots a_n$ is a *subword* of a word w if, and only if, $w \in \Sigma^* a_1 \Sigma^* \dots \Sigma^* a_n \Sigma^*$. The aim of this exercise is to prove the following characterization of finite $\mathcal{J}$-trivial monoids, due to I. Simon:

Simon's Lemma. A finite monoid M is $\mathcal{J}$-trivial if, and only if, there exists $m \in \mathbb{N}$ such that, for any words $u, v \in M^*$, if u and v contain exactly the same subwords of length $\leq m$, then $(u)_M = (v)_M$.

a. For the sufficiency of the condition, assume such an m exists. Show directly from the definitions that M must in particular be $\mathcal{J}$-trivial.

b. (*) Suppose M is $\mathcal{J}$-trivial, and write n for the maximal length of a strict $\mathcal{J}$-chain in the finite monoid M. Prove that if u and v are words in M^* that contain exactly the same subwords of length $\leq 2n-1$, then necessarily $(u)_M = (v)_M$.

c. From the previous item, deduce the necessity of the condition in Simon's Lemma.

d. Show that a finite monoid M is $\mathcal{J}$-trivial if, and only if, M is aperiodic and there exists $n \in \mathbb{N}$ such that for every $x, y \in M$, we have $(xy)^n = (yx)^n$.

Exercise 2. This exercise indicates a proof that the languages that are recognizable by a finite $\mathcal{J}$-trivial monoid are exactly those that are piecewise testable, i.e., a finite Boolean combination of languages that are upward closed in the subword ordering, $\preceq$. We use Higman's Lemma (see Exercise 6 of the previous problem sheet) and Simon's Lemma (see the previous exercise). Let Σ be a finite alphabet.

a. Deduce from Higman's Lemma that a $\preceq$-upward closed language is a finite union of languages of the form $\Sigma^* a_1 \Sigma^* \dots \Sigma^* a_n \Sigma^*$, where $a_1, \dots, a_n \in \Sigma$.

b. Show that any language of the form $\Sigma^* a_1 \Sigma^* \dots \Sigma^* a_n \Sigma^*$ can be recognized by a finite $\mathcal{J}$-trivial monoid.

c. Conversely, let M be a finite $\mathcal{J}$-trivial monoid and $\Sigma \subseteq M$. Using Simon's Lemma, show that, for any $m \in M$, the set of words $w \in \Sigma^*$ such that $(w)_M = m$ is upward closed in the subword ordering.

Boolean algebras and profinite monoids

Exercise 3. Let X be a set.

a. Let A be a finite Boolean subalgebra of $\mathcal{P}(X)$. Show that the atoms of A form a partition of X.

b. Show that there is a bijection ϕ between the finite index equivalence relations on X and the finite Boolean subalgebras of $\mathcal{P}(X)$, which satisfies $\equiv_1 \subseteq \equiv_2$ if, and only if, $\phi(\equiv_2) \subseteq \phi(\equiv_1)$.

Exercise 4. This exercise is a brief introduction to a duality theorem first proved by M. H. Stone in the 1930's. We only consider Boolean algebras here, although Stone's theorem was for general distributive lattices.
Let A be a Boolean algebra. On the set X of *characters* on A, i.e., the set of homomorphisms from A to the two-element algebra, define the topology τ generated by the sets

$$\widehat{a} := \{x \in X_A \colon x(a) = 1\}.$$

a. Recall that a topological space X is called *Boolean* if it is compact, Hausdorff, and the clopen sets form a basis for the topology. Prove that τ is a Boolean topology in which the clopen sets are exactly the sets of the form $\widehat{a}$ for $a \in A$.

b. Let B be a Boolean algebra with space of characters Y. For any continuous function $f \colon Y \to X$, write $h_f \colon A \to B$ for the function sending $a \in A$ to the element $b \in B$ such that $\widehat{b} = f^{-1}(\widehat{a})$. Prove that $f \mapsto h_f$ is a bijection between continuous functions from Y to X and homomorphisms from A to B. Deduce from this that the categories of Boolean algebras and of Boolean spaces are dually equivalent.

c. How does the bijection from the previous item restrict to *injective* homomorphisms between Boolean algebras? Explain how this generalizes Exercise 3 above.

Exercise 5. Prove that a topological space is Boolean if, and only if, it is a projective limit of finite discrete spaces in the category of topological spaces. Thus, Boolean spaces are "profinite sets".

Exercise 6. Let M be a profinite monoid. Prove that for any $x \in M$, there is exactly one idempotent element in $\overline{\{x^n \ : \ n \in \mathbb{N}\}}$.

Exercise 7. Let K a clopen set in a topological monoid M whose topology is Boolean. Define the equivalence relation $\sim_K$ on M by: $u \sim_K v$ if, and only if, for all $x, y \in M$, $xuy \in K$ iff $xvy \in K$.

a. Prove that $\sim_K$ is a congruence.

b. Prove that, for any $u \in M$, $[u]_{\sim_K}$ is closed.

c. Prove that, for any $u \in M$, $[u]_{\sim_K}$ is open.

d. Conclude that $\sim_K$ has finite index, and that the quotient map $M \twoheadrightarrow M/{\sim_K}$ is continuous.

Exercise 8. Compute the syntactic monoids of the languages $(aa)^*$, $\Sigma^* a \Sigma^*$ and $(ab)^*$, where $\Sigma = \{a, b\}$.

Exercise 9. This exercise completes the proof of Eilenberg's Theorem given in the lecture.

a. Prove that if M' is a submonoid of M, and L is recognized by M', then L is also recognized by M.

b. Let $\mathbb{V}$ a variety of monoids. Prove that if L is recognized by some monoid M in $\mathbb{V}$, then the syntactic monoid of L is in $\mathbb{V}$.

c. Now suppose that M is a finite monoid such that any language L recognized by M has syntactic monoid in $\mathbb{V}$. In particular, for each $m \in M$, the syntactic monoid, M_m, of the language $L_m := \{w \in M^* : (w)_M = m\}$, is in $\mathbb{V}$. Prove that the function $f \colon M \to \prod_{m \in M} M_m$ defined by sending $x \in M$ to $([x]_{\equiv_{L_m}})_{m \in M}$ is an injective homomorphism, and conclude from this that $M \in \mathbb{V}$.