Finite \( J \)-trivial monoids

**Exercise 1.** Recall that \( a_1 \ldots a_n \) is a subword of a word \( w \) if, and only if, \( w \in \Sigma^*a_1\Sigma^* \ldots \Sigma^*a_n\Sigma^* \). The aim of this exercise is to prove the following characterization of finite \( J \)-trivial monoids, due to I. Simon:

**Simon’s Lemma.** A finite monoid \( M \) is \( J \)-trivial if, and only if, there exists \( m \in \mathbb{N} \) such that, for any words \( u, v \in M^* \), if \( u \) and \( v \) contain exactly the same subwords of length \( \leq m \), then \( (u)_M = (v)_M \).

a. For the sufficiency of the condition, assume such an \( m \) exists. Show directly from the definitions that \( M \) must in particular be \( J \)-trivial.

b. (*) Suppose \( M \) is \( J \)-trivial, and write \( n \) for the maximal length of a strict \( J \)-chain in the finite monoid \( M \). Prove that if \( u \) and \( v \) are words in \( M^* \) that contain exactly the same subwords of length \( \leq 2n - 1 \), then necessarily \( (u)_M = (v)_M \).

c. From the previous item, deduce the necessity of the condition in Simon’s Lemma.

d. Show that a finite monoid \( M \) is \( J \)-trivial if, and only if, \( M \) is aperiodic and there exists \( n \in \mathbb{N} \) such that for every \( x, y \in M \), we have \((xy)^n = (yx)^n\).

**Exercise 2.** This exercise indicates a proof that the languages that are recognizable by a finite \( J \)-trivial monoid are exactly those that are piecewise testable, i.e., a finite Boolean combination of languages that are upward closed in the subword ordering, \( \preceq \). We use Higman’s Lemma (see Exercise 6 of the previous problem sheet) and Simon’s Lemma (see the previous exercise). Let \( \Sigma \) be a finite alphabet.

a. Deduce from Higman’s Lemma that a \( \preceq \)-upward closed language is a finite union of languages of the form \( \Sigma^*a_1\Sigma^* \ldots \Sigma^*a_n\Sigma^* \), where \( a_1, \ldots, a_n \in \Sigma \).

b. Show that any language of the form \( \Sigma^*a_1\Sigma^* \ldots \Sigma^*a_n\Sigma^* \) can be recognized by a finite \( J \)-trivial monoid.

c. Conversely, let \( M \) be a finite \( J \)-trivial monoid and \( \Sigma \subseteq M \). Using Simon’s Lemma, show that, for any \( m \in M \), the set of words \( w \in \Sigma^* \) such that \((w)_M = m \) is upward closed in the subword ordering.

**Boolean algebras and profinite monoids**

**Exercise 3.** Let \( X \) be a set.

a. Let \( A \) be a finite Boolean subalgebra of \( \mathcal{P}(X) \). Show that the atoms of \( A \) form a partition of \( X \).

b. Show that there is a bijection \( \phi \) between the finite index equivalence relations on \( X \) and the finite Boolean subalgebras of \( \mathcal{P}(X) \), which satisfies \( \equiv_1 \subseteq \equiv_2 \) if, and only if, \( \phi(\equiv_2) \subseteq \phi(\equiv_1) \).
Exercise 4. This exercise is a brief introduction to a duality theorem first proved by M. H. Stone in the 1930’s. We only consider Boolean algebras here, although Stone’s theorem was for general distributive lattices.

Let $A$ be a Boolean algebra. On the set $X$ of characters on $A$, i.e., the set of homomorphisms from $A$ to the two-element algebra, define the topology $\tau$ generated by the sets

$$a := \{ x \in X_A : x(a) = 1 \}.$$

a. Recall that a topological space $X$ is called Boolean if it is compact, Hausdorff, and the clopen sets form a basis for the topology. Prove that $\tau$ is a Boolean topology in which the clopen sets are exactly the sets of the form $a$ for $a \in A$.

b. Let $B$ be a Boolean algebra with space of characters $Y$. For any continuous function $f: Y \to X$, write $h_f: A \to B$ for the function sending $a \in A$ to the element $b \in B$ such that $\hat{b} = f^{-1}(\hat{a})$. Prove that $f \mapsto h_f$ is a bijection between continuous functions from $Y$ to $X$ and homomorphisms from $A$ to $B$. Deduce from this that the categories of Boolean algebras and of Boolean spaces are dually equivalent.

c. How does the bijection from the previous item restrict to injective homomorphisms between Boolean algebras? Explain how this generalizes Exercise 3 above.

Exercise 5. Prove that a topological space is Boolean if, and only if, it is a projective limit of finite discrete spaces in the category of topological spaces. Thus, Boolean spaces are “profinite sets”.

Exercise 6. Let $M$ be a profinite monoid. Prove that for any $x \in M$, there is exactly one idempotent element in $\{x^n : n \in \mathbb{N}\}$.

Exercise 7. Let $K$ a clopen set in a topological monoid $M$ whose topology is Boolean. Define the equivalence relation $\sim_K$ on $M$ by: $u \sim_K v$ if, and only if, for all $x, y \in M$, $xuv \in K$ iff $xvy \in K$.

a. Prove that $\sim_K$ is a congruence.

b. Prove that, for any $u \in M$, $[u]_{\sim_K}$ is closed.

c. Prove that, for any $u \in M$, $[u]_{\sim_K}$ is open.

d. Conclude that $\sim_K$ has finite index, and that the quotient map $M \to M/\sim_K$ is continuous.

Exercise 8. Compute the syntactic monoids of the languages $(aa)^*, \Sigma^*a\Sigma^*$ and $(ab)^*$, where $\Sigma = \{a, b\}$.

Exercise 9. This exercise completes the proof of Eilenberg’s Theorem given in the lecture.

a. Prove that if $M'$ is a submonoid of $M$, and $L$ is recognized by $M'$, then $L$ is also recognized by $M$.

b. Let $\mathbb{V}$ a variety of monoids. Prove that if $L$ is recognized by some monoid $M$ in $\mathbb{V}$, then the syntactic monoid of $L$ is in $\mathbb{V}$.
c. Now suppose that $M$ is a finite monoid such that any language $L$ recognized by $M$ has syntactic monoid in $\mathbb{V}$. In particular, for each $m \in M$, the syntactic monoid, $M_m$, of the language $L_m := \{ w \in M^* : (w)_M = m \}$, is in $\mathbb{V}$. Prove that the function $f : M \to \prod_{m \in M} M_m$ defined by sending $x \in M$ to $([x]_{\equiv_{l,m}})_{m \in M}$ is an injective homomorphism, and conclude from this that $M \in \mathbb{V}$. 