Logique modale

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These are lecture notes for the part of the L3 course *Logique* at ENS Paris-Saclay in which we talk about modal logic. These notes are for a large part based on the first two chapters of the textbook [1], which contains a lot more.

1 Syntax and semantics

Definition 1. We fix a set of propositional variables P. A *formula* of modal propositional logic is an expression generated by the following grammar:

$$\varphi ::= \top \mid p \mid \varphi \vee \varphi \mid \neg\varphi \mid \Box\varphi \ .$$

We call a formula *Boolean* if it contains no $\Box$.

Remark 2. The study of modal logic in fact covers a more general syntax, where one can have several modalities, instead of just one, and the modalities can be n-ary for $n > 1$, instead of unary. We do not treat this here. It is also possible to work over a *positive* instead of a Boolean base, meaning that one takes $\vee, \wedge, \top, \bot$ as basic non-modal connectives. We also do not treat this here.

Recall that a Boolean formula is *valid* if it evaluates to 1 under all two-valued valuations of propositions. We extend this notion of validity to modal formulas.

Definition 3. A *frame* is a pair (W, R) where W is a set and $R \subseteq W \times W$ is a binary relation.
A *model* is a triple (W, R, V), where (W, R) is a frame and $V : W \to \mathcal{P}(\mathsf{P})$ is a function.
A *pointed model* is a quadruple (W, R, V, w) where (W, R, V) is a model and $w \in W$.
We define the satisfaction relation $\models$. Let $\mathcal{M} = (W, R, V)$ be a model and $w \in W$. We define:

- $\mathcal{M}, w \models \top$ always;
- $\mathcal{M}, w \models p$ if, and only if, $p \in V(w)$;
- $\mathcal{M}, w \models \varphi \vee \psi$ if, and only if, $\mathcal{M}, w \models \varphi$ or $\mathcal{M}, w \models \psi$;
- $\mathcal{M}, w \models \neg\varphi$ if, and only if, it is not the case that $\mathcal{M}, w \models \varphi$;
- $\mathcal{M}, w \models \Box\varphi$ if, and only if, for all $v \in W$ such that wRv, $\mathcal{M}, v \models \varphi$.

We also write $\mathcal{M} \models \varphi$ if, for all $w \in W$, we have $\mathcal{M}, w \models \varphi$.
For a frame $\mathcal{F} = (W, R)$, we write $\mathcal{F} \models \varphi$ if, for all $V : W \to \mathcal{P}(\mathsf{P})$, $(W, R, V) \models \varphi$.

Example 4. Let K be the formula

$$\Box(p \to q) \to (\Box p \to \Box q).$$

We show that $\mathcal{M} \models \mathsf{K}$ for any model $\mathcal{M}$. Let w be an arbitrary node in $\mathcal{M}$ and suppose that $\mathcal{M}, w \models \Box(p \to q)$ and $\mathcal{M}, w \models \Box p$. Let v in $\mathcal{M}$ be arbitrary such that wRv. Then $\mathcal{M}, v \models p \to q$ and $\mathcal{M}, v \models p$. Thus, $\mathcal{M}, v \models q$, as required.

Definition 5. We introduce an abbreviation:

$$\Diamond\varphi := \neg\Box\neg\varphi\ .$$

We also use the usual propositional abbreviations: $\varphi \wedge \psi$ for $\neg(\neg\varphi \vee \neg\psi)$, $\varphi \to \psi$ for $\neg\varphi \vee \psi$, $\varphi \leftrightarrow \psi$ for $(\varphi \to \psi) \wedge (\psi \to \varphi)$ and $\bot$ for $\neg\top$.

Remark 6. We have

$$\mathcal{M}, w \models \Diamond\varphi \text{ if, and only if, there exists } v \in W \text{ such that } \mathcal{M}, v \models \varphi\ .$$

Remark 7. Everything we do here is over a *classical* base. Intuitionistic versions of modal logic exist and this is a currently fairly popular object of study, but we do not treat it in this course.

Definition 8. *Local semantic consequence* is defined as follows. Let Γ be a set of formulas and φ a formula. We write $\Gamma \models \varphi$ if, and only if, for all pointed models $(\mathcal{M}, w)$, if $\mathcal{M}, w \models \gamma$ for all $\gamma \in \Gamma$, then $\mathcal{M}, w \models \varphi$.
Global semantic consequence is the following variant: we write $\Gamma \models^g \varphi$ if, and only if, for all models $\mathcal{M}$, if $\mathcal{M} \models \gamma$ for all $\gamma \in \Gamma$, then $\mathcal{M} \models \varphi$.

Remark 9. If $\Gamma \models \varphi$, then $\Gamma \models^g \varphi$.

Example 10. We have $p \models^g \Box p$ but it is not the case that $p \models \Box p$.

2 A Hilbert system

Definition 11. A *normal modal logic* is a set Λ of formulas such that

1. for any Boolean formula φ, if φ is valid then φ is in Λ;
2. the formula K is in Λ;
3. if φ and $\varphi \to \psi$ are in Λ, then ψ is in Λ;
4. if φ is in Λ, then $\Box\varphi$ is in Λ;
5. if $\varphi(p_1, \dots, p_n)$ is in Λ and $\sigma_1, \dots, \sigma_n$ are any formulas, then $\varphi(\sigma_1, \dots, \sigma_n)$ is in Λ.

In other words, a normal modal logic is one which contains (1) **CPC**, (2) the axiom K, and is closed under (3) modus ponens, (4) necessitation, and (5) uniform substitution.
The *smallest normal modal logic* is called **K**.

Note that **K** is well-defined because normal modal logics are closed under intersections.

Definition 12. Let φ be a formula. A **K**-*proof* is a non-empty finite sequence $\varphi_0, \ldots, \varphi_n$ such that the following hold:

- $\varphi_n = \varphi$, and
- for each $0 \leq i \leq n$, φ_i is either a valid Boolean formula, or the axiom K, or obtainable from formulas in $\{\varphi_j \mid j < i\}$ by applying modus ponens, necessitation, or uniform substitution.

Proposition 13. *Let φ be a formula. Then φ is in* **K** *if, and only if, there exists a* **K***-proof of φ.*

Proof. Denote by $\mathbf{K}'$ the set of formulas φ for which a **K**-proof exists. Note that $\mathbf{K}'$ is a normal modal logic, and thus contains **K**. On the other hand, if Λ is any normal modal logic, then one shows by induction that, for any $n \in \mathbb{N}$, if φ admits a **K**-proof of length $\leq n$, then $\varphi \in \Lambda$. In particular, taking $\Lambda := \mathbf{K}$, we get that $\mathbf{K}' \subseteq \mathbf{K}$. □

Definition 14. We write $\vdash_{\mathbf{K}} \varphi$ if the conditions in Proposition 13 hold for φ.

Remark 15. The formula $(\Box p \wedge \Box q) \leftrightarrow \Box(p \wedge q)$ is in **K**. (Exercise.)

Definition 16. Let φ and ψ be formulas. We say that φ *implies* ψ (in **K**) if $\vdash_{\mathbf{K}} \varphi \to \psi$; we write $\varphi \preceq_{\mathbf{K}} \psi$ or also just $\varphi \preceq \psi$. We say that φ and ψ are *equivalent* (in **K**) if $\vdash_{\mathbf{K}} \varphi \leftrightarrow \psi$. In this case, we write $\varphi \equiv_{\mathbf{K}} \psi$ or also just $\varphi \equiv \psi$.

Remark 17. If we would have a **K**-proof of p, then by necessitation we would also have a **K**-proof of $\Box p$. However, we cannot reason as in natural deduction and 'introduce an arrow' to obtain from this reasoning a **K**-proof of $p \to \Box p$. In fact, no **K**-proof of p should exist, because then by the uniform substitution rule we would also have a **K**-proof of $\bot$. It will follow from soundness (Theorem 19) that no such **K**-proof exists.

Remark 18. Definition 12 is sometimes called a *Hilbert-style proof system*, as opposed to the Gentzen-style systems of natural deduction and **LK**, **LJ**, which we saw earlier in this course. The advantages of a Hilbert-style system are that it is short to define, easy to remember, and that the definition of the notion of 'proof' in such a system only requires a notion of 'list' in the meta-theory, rather than trees, which can simplify some induction arguments. We will see below that the connection with algebra is also easily made. On the other hand, it is not clear in general how to obtain decidability of the membership problem for a logic when it is given by a Hilbert-style system.

Theorem 19 (Soundness)**.** *Let φ be a formula. If $\vdash_{\mathbf{K}} \varphi$, then, for any frame $\mathcal{F}$, we have $\mathcal{F} \models \varphi$.*

Proof. We show by induction on the length of a proof of φ that, for any pointed model $\mathcal{M}, w$, we have $\mathcal{M}, w \models \varphi$.
Let $n \geq 0$ and suppose that $\pi = (\varphi_0, \ldots, \varphi_n)$ is a **K**-proof of φ, so $\varphi_n = \varphi$. Let $\mathcal{M}, w$ be a pointed model. We distinguish cases according to the justification for φ_n in the proof π.
If φ_n is a valid Boolean formula, then we certainly have $\mathcal{M}, w \models \varphi$.
If φ_n is the axiom K, then $\mathcal{M}, w \models \varphi_n$ by Example 4.
Otherwise, φ_n is obtainable by one of the three rules. (In particular, we must then have $n > 0$.) Suppose, for instance, that φ_n is obtained by the necessitation rule from φ_i for some $0 \leq i < n$, so that $\varphi_n = \Box\varphi_i$. The length $i + 1$ prefix of π is a **K**-proof of φ_i, so the induction hypothesis in particular yields that $\mathcal{M}, v \models \varphi_i$ for all R-successors v of w. Thus, $\mathcal{M}, w \models \Box\varphi_i$, which is φ_n.
The proofs for the cases of modus ponens and uniform substitution are left as an exercise. □

Definition 20. Let Γ be a set of formulas. The smallest normal modal logic that contains Γ is called the *normal modal logic generated by* Γ or *the extension of* $\mathbf{K}$ *by* Γ. We denote it by $\mathbf{K}\Gamma$ (with some exceptions, see below).

Remark 21. The Hilbert-style definition of $\mathbf{K}$-proof can be extended to a definition of $\mathbf{K}\Gamma$-proof by allowing in addition that a proof contains any element of Γ. The same proof as the one given above for Proposition 13 then shows that the formulas in $\mathbf{K}\Gamma$ are exactly those for which a $\mathbf{K}\Gamma$-proof exists. (Exercise.)

Example 22. We list a few formulas that are commonly used as axioms, and their commonly used names:

- (4) $\Box p \to \Box\Box p$
- (T) $\Box p \to p$
- (B) $p \to \Box\Diamond p$
- (L) $\Box(\Box p \to p) \to \Box p$
- (D) $\Box p \to \Diamond p$

We write $\mathbf{K4}$ instead of $\mathbf{K}\{4\}$, etc. $\mathbf{K}\{4, T\}$ is called $\mathbf{S4}$ and $\mathbf{K}\{4, T, B\}$ is called $\mathbf{S5}$. $\mathbf{KL}$ is also known as $\mathbf{GL}$. We do not comment on the history or idiosyncracy of these names.

Remark 23. Some references, like [1], give $\Diamond$-versions of some of the axioms. It is a matter of Boolean calculation to translate between the two versions. We do the calculation just for (T):

$$\Box p \to p \equiv \neg\Diamond\neg p \to p \equiv \neg p \to \Diamond\neg p$$

and by applying the substitution $\neg p \mapsto p$, we get the diamond version of (T):

$$p \to \Diamond p\ .$$

It is an instructive exercise to translate (L) into its $\Diamond$-form. Note that we use in this remark that $\equiv$ is an equivalence relation, which we prove below.

Remark 24. $\mathbf{KD}$ can alternatively be axiomatized by the formula $\Diamond\top$. (Exercise.)

3 Bisimulation-invariance

Definition 25. A *bisimulation* from a model $\mathcal{M}_1 = (W_1, R_1, V_1)$ to a model $\mathcal{M}_2 = (W_2, R_2, V_2)$ is a relation $Z \subseteq W_1 \times W_2$ such that the following hold for any $(w_1, w_2) \in Z$:

1. $V_1(w_1) = V_2(w_2)$;
2. for any $u \in W_1$, if $w_1 R_1 u$, then there exists $v \in W_2$ such that $w_2 R_2 v$ and $(u, v) \in Z$;
3. for any $v \in W_2$, if $w_2 R_2 v$, then there exists $u \in W_1$ such that $u R_1 v$ and $(u, v) \in Z$.

For pointed models, we say that $\mathcal{M}_1, w_1$ is *bisimilar to* $\mathcal{M}_2, w_2$ if there exists a bisimulation from $\mathcal{M}_1$ to $\mathcal{M}_2$ which contains (w_1, w_2).

Proposition 26. *Modal formulas are invariant under bisimulation. That is, if $\mathcal{M}_1, w_1$ is bisimilar to $\mathcal{M}_2, w_2$, then for any modal formula φ, we have $\mathcal{M}_1, w_1 \models \varphi$ if, and only if, $\mathcal{M}_2, w_2 \models \varphi$.*

Proof. By induction on φ. The case of propositional variables holds by (1) in Definition 25, and the cases $\vee$ and $\neg$ are simple. For the case $\Box$, suppose that $\mathcal{M}_1, w_1 \models \Box\psi$. In order to show that $\mathcal{M}_2, w_2 \models \Box\psi$, let $v \in W_2$ be arbitrary such that $w_2 R_2 v$. By (3) in Definition 25, pick $u \in W_1$ such that $w_1 R_1 u$ and $(u, v) \in Z$. Since $\mathcal{M}_1, w_1 \models \Box\psi$, we have $\mathcal{M}_1, u \models \psi$. By the induction hypothesis, $\mathcal{M}_2, v \models \psi$. The proof of the converse direction is the same, this time using (2) in Definition 25. $\Box$

References

[1] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*, volume 53 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 2001.