Unification, duality, and de Bruijn graphs

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LAC Seminar

Università degli Studi di Milano

A question

You forgot your digicode. The keypad lets you enter a sequence of any length. As soon as the correct code appears, the door opens. What sequence do you enter?

The start of an answer

0001AAA0002AA010AAA100AA110011A0012A0021A01110101A101210022A0A0A1A020A01220003AA0211020102A102210A1111200A120112AAA20023
A0031A030A02220AA2102310032A02320033A013A0A2A0311030103A1032103310131OA211A1A121AA122A1122211230004AA03220202120 2A201A20
2330013201330113 00A220302031203A203320A230A130123A113AA131A132A1232113221212A2123311133A1A2A13321A221302131213A213330AA3
0034A0041A040A03331AA2222302231 1A231223AAA3103410042A0411040104A10421043A0342004310 44A014A024A0A3A04220402041204A2043203
43004410141024 10A311340005AA0433030313032303A301A302A3034400142014301440114002420243024402140 12400A3204420A3304030413042
304A304430A340134A114AA1402240A141A142A143A1342114221412 14A2143213431143313132313A312A3134411144A124A1A3A144212421A32144
312431A3314031413142314A314440A2402340AA40045A0051A050A0444122411244 1A24123411A341AA322A2A232AA233A223332234A224AA242A24
3A2343224332323A3234422244A2A3A24432A3324032413242324A324442AA3333403341334 22A342334AAA4104510052A0511050105A10521053A05
220502051205A20531054A045200532054105 5A015A025A035A0A4A0533050305130523 05A3054204530054304540055101510251035 10A411450006

Aim of the talk

The aim of this talk is to explain what de Bruijn graphs are, and how they relate to a problem of unifiability in logic.

We will see that the connection is made by Stone duality, a general theory for linking syntax and semantics.

Overview

De Bruijn graphs

Unifiability

Stone duality

de Bruijn graphs

The de Bruijn graph $B_d(\Sigma)$ of order d over alphabet Σ is the deterministic automaton that 'remembers the last d letters'.

de Bruijn graphs

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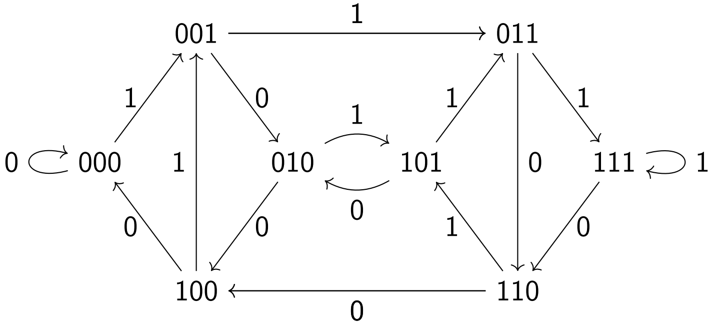
For example, when $d = 3$ and $\Sigma = \{0, 1\}$:



Named for N. G. de Bruijn (1946), also invented by I. J. Good (1946), implicit in C. Flye Sainte-Marie (1894), and also in ancient Sanskrit prosody; see Knuth, vol. 4A, 7.2.1.7, p. 489.

Graphs

Fix a finite alphabet Σ.

Definition

A Σ-edge-labeled directed graph consists of:

- a set of vertices V,
- for each $a \in \Sigma$, an edge relation $\xrightarrow{a} \subseteq V^2$.

Example

For any $d \geq 1$, the de Bruijn graph of order d, $B_d(\Sigma)$, has

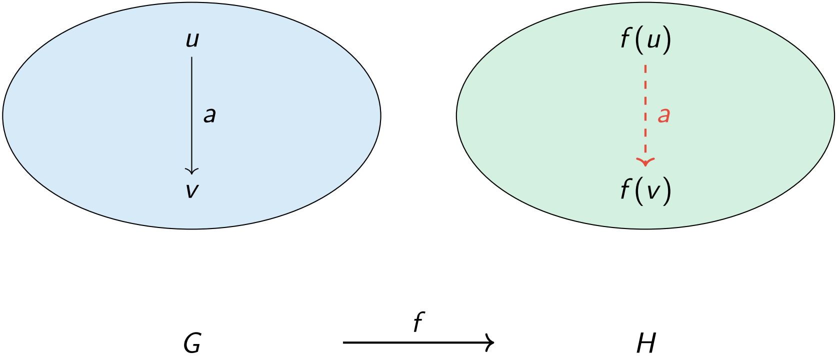
- set of vertices Σ^d,
- for each $a \in \Sigma$ and $w \in \Sigma^d$, a labeled edge

  $$w \xrightarrow{a} w'a$$

  where w' is the length $d-1$ suffix of w.

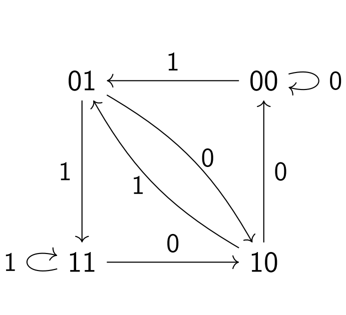
Graph homomorphisms

A homomorphism from a graph G to a graph H is a function from V_G to V_H that preserves labeled edges.



Homomorphism from a de Bruijn graph

Is there a homomorphism from $B_2(\{0,1\})$ to the graph G?



$B_2(\{0,1\})$ G

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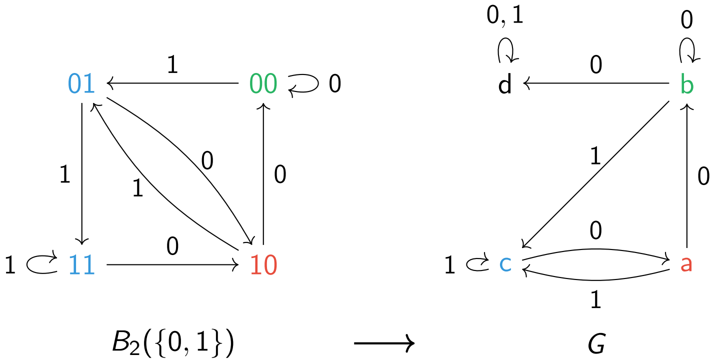

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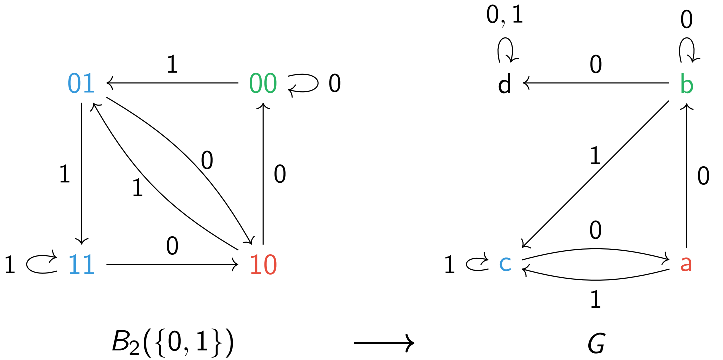

- The codomain graph may fail to be deterministic in general.
- There may be more than one homomorphism.

Homomorphism from a de Bruijn graph?

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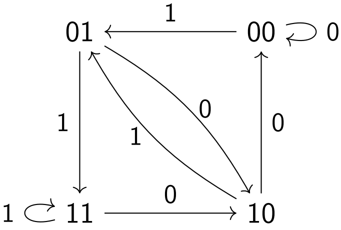
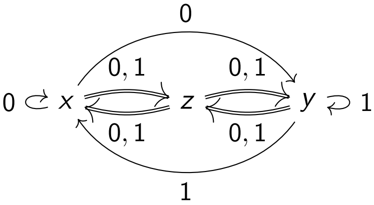


$B_2(\{0,1\})$



H

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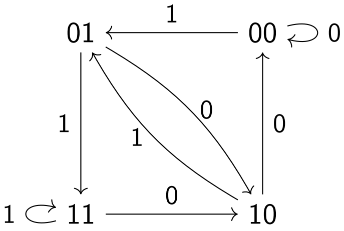

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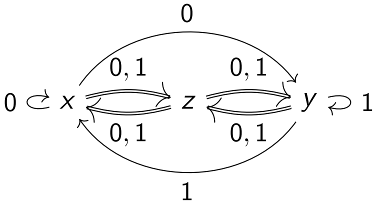

H

Is there a homomorphism from $B_d(\{0,1\})$ to H for some d?

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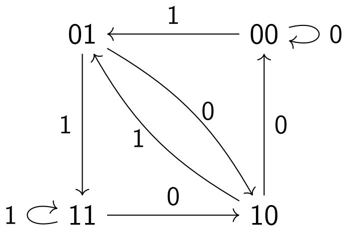

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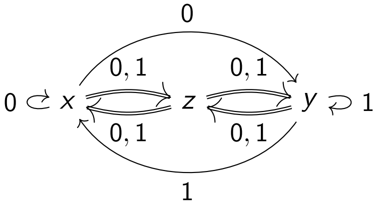

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How can you be sure that there is none? ▸ Answer for $d = 2$

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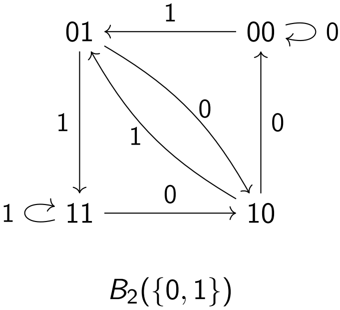

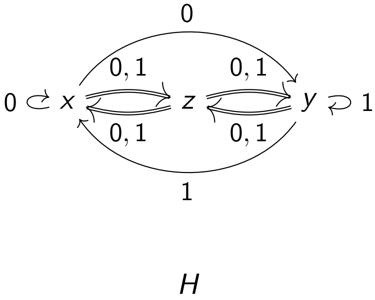

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How to decide this for a general graph? ▸ Another example

The de Bruijn graph mapping problem

Fix a finite alphabet Σ.

Problem (de Bruijn graph mapping)

INPUT. A finite Σ-edge-labeled directed graph G.

OUTPUT.

- a number $d \geq 1$ and a homomorphism $B_d \to G$, or
- 'impossible' if none exists.

We usually consider the *surjective* version of the problem, which is at least as hard. Explanation

Overview

De Bruijn graphs

Unifiability

Stone duality

The unifiability problem

We arrived at this problem because of a problem in temporal logic:

Problem (Unifiability in temporal logic of next)

INPUT. A formula φ in the temporal logic of next.

OUTPUT.

- a unifier for φ, or
- 'impossible' if none exists.

Unifiers defined

Two sets: variables $V = \{x, y, \dots\}$, constants $C = \{p_1, p_2, \dots\}$.

A formula is an expression built from these with $\vee$, $\neg$, $\bot$, and $\mathbf{X}$.

The depth of a formula is the maximum nesting depth of $\mathbf{X}$ in it.

A unifier of a formula $\varphi(x, y, \dots)$ is a substitution

$$x \mapsto \sigma_x, y \mapsto \sigma_y, \dots,$$

where, for each $x \in V$, σ_x is a formula, such that

$$\varphi(x \mapsto \sigma_x, y \mapsto \sigma_y, \dots)$$

is a valid formula.

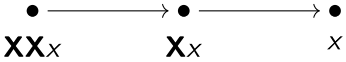
Validity defined

There are two equivalent views on validity in the logic of next:

- Syntactic: A formula φ is valid if the equation $\varphi \equiv \top$ follows from the rules of Boolean logic, together with

$$\mathbf{X}(\varphi_1 \vee \varphi_2) \equiv \mathbf{X}\varphi_1 \vee \mathbf{X}\varphi_2, \quad \mathbf{X}(\neg\varphi) \equiv \neg\mathbf{X}\varphi, \text{ and } \mathbf{X}\top \equiv \top.$$

- Semantic: A formula φ is valid if it evaluates to true in all deterministic transition systems. $\mathbf{X}\varphi$ is evaluated as 'next φ':



The other connectives are evaluated locally at each state.

The equivalence is called the completeness theorem for the logic.

Examples

Is this formula unifiable?

$$[x \leftrightarrow (\neg \mathbf{X} p \wedge \mathbf{XX}(x \vee y))] \wedge [y \leftrightarrow (x \rightarrow p)]$$

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The (unique, in this case) unifier: $x \mapsto \neg \mathbf{X} p$, $y \mapsto p \vee \mathbf{X} p$.

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Negative example: $x \leftrightarrow \neg \mathbf{X} x$ does not have a unifier. ▸ Explanation

Unifiers and de Bruijn graph homomorphisms

Let φ be a formula in the logic of next.

We construct a finite graph $G(\varphi)$ with edge-labels in $\Sigma := 2^C$.
(This can take up to $\mathcal{O}(\exp(|\varphi|))$ time.)

Theorem (v.G. & Marti, 2023)

The set of $\equiv$-classes of depth $\leq d$ ground unifiers of φ

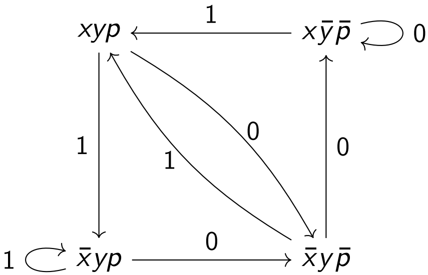
is in bijection with

the set of homomorphisms from $B_d(2^C)$ to $G(\varphi)$.

The depth of σ is the maximum depth of the formulas σ_x.
A unifier σ is ground if no variables are used in any σ_x.
Unifiers σ and σ' are equivalent if $\sigma_x \equiv \sigma'_x$ for all x.

Example of the graph associated to a formula

$$\varphi \ : \ [x \leftrightarrow \neg \mathbf{X} p] \wedge [y \leftrightarrow (x \rightarrow p)]$$



$G(\varphi)$

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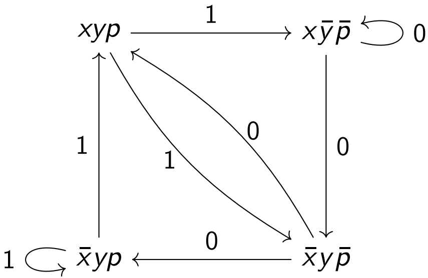

$G(\varphi)$ reversed

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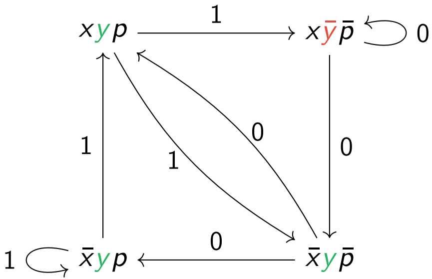

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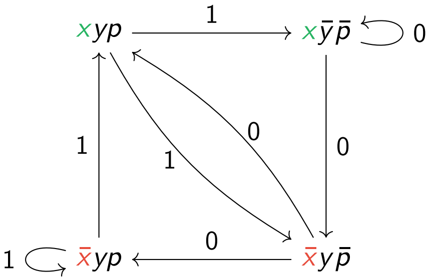

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Example of the graph associated to a formula

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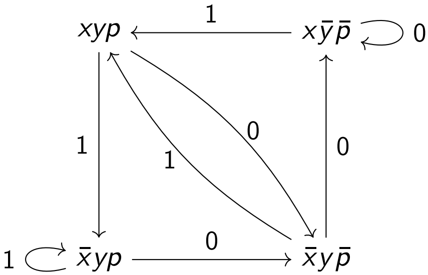


$G(\varphi)$

The graph $G(\varphi)$ is an image of B_2, so φ is unifiable.

The more general picture: Equational Unification

Let $\mathbf{V}$ be a variety of algebras in a signature τ.

Problem (Unifiability in $\mathbf{V}$)

Given a finite set E of τ-equations

$$s(x_1, \ldots, x_n) \approx t(x_1, \ldots, x_n) \ ,$$

does there exist a substitution $x_i \mapsto u_i$ such that

$$\mathbf{V} \models s(u_1, \ldots, u_n) \approx t(u_1, \ldots u_n)$$

for each $s \approx t$ in E?

Unifiability and finitely presented algebras

Problem (Unifiability in $\mathbf{V}$, algebraic version)

Given a finitely presented algebra A in $\mathbf{V}$, does there exist a homomorphism from A to $F_{\mathbf{V}}(\emptyset)$?

A finite set of τ-equations E gives a finitely presented algebra

$$A := F_{\mathbf{V}}(x_1, \ldots, x_n)/\langle E \rangle_{\mathrm{con}} \ ,$$

and a ground unifier is a homomorphism $A \to F_{\mathbf{V}}(\emptyset)$.

Idea: It is sometimes useful to dualize this problem and to use stratification of the free algebras.

(Ghilardi 1999, Ghilardi & Zawadowski 2002)

An open problem: Unifiability for modal algebras

A modal algebra is a tuple $(B, \Diamond)$, with B a Boolean algebra and a function $\Diamond : B \to B$ such that, for all $a, b \in B$,

$$\Diamond(a \vee b) = \Diamond a \vee \Diamond b \text{ and } \Diamond\bot = \bot.$$

The unifiability problem for modal algebras is open.

Its unification type is nullary (Jerabek 2011) and slight extensions have undecidable unification (Wolter & Zakharyashev 2006).

Our result: Unifiability for pointed dMA's

A modal algebra $(B, \mathbf{X})$ is deterministic if, for every $a \in B$,

$$\mathbf{X}\neg a = \neg \mathbf{X} a \ .$$

A pointed dMA is a tuple $(B, \mathbf{X}, c)$, where $(B, \mathbf{X})$ is a deterministic modal algebra, and $c \in B$.

Theorem (Marti, v.G., Sweering 2024)

Unifiability is decidable for pointed dMA's.

Proof.

Duality + decide the de Bruijn graph mapping problem! $\square$

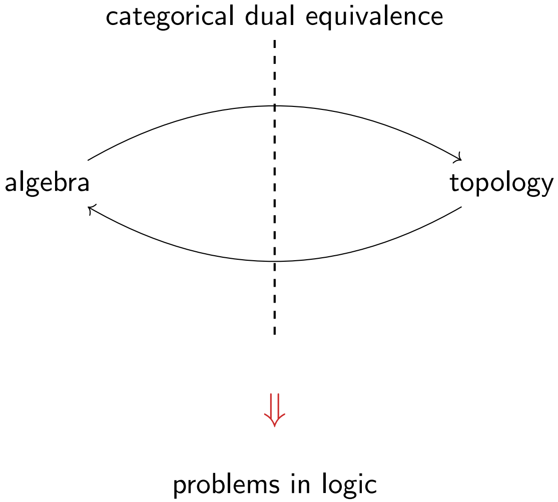
Overview

De Bruijn graphs

Unifiability

Stone duality

Stone duality: The big picture



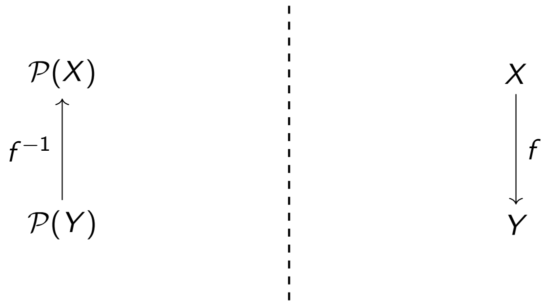
A textbook treatment

M. Gehrke and SvG, Topological Duality for Distributive Lattices, Cambridge University Press (2024).

Boolean algebras

For any set X, the power set $\mathcal{P}(X)$ is a Boolean algebra, i.e., its operations $\cup$, $\cap$ and $()^{\mathrm{c}}$ precisely obey the rules of $\vee$, $\wedge$, and $\neg$.

Moreover, any function $f: X \to Y$ gives rise to a Boolean algebra homomorphism in the other direction:



Thus, we have a functor $\mathcal{P}: \mathrm{Set} \to \mathrm{BA}^{\mathrm{op}}$.

Stone representation for Boolean algebras

Theorem (Stone, 1936)

For any Boolean algebra B, the homomorphism

$$\widehat{(-)}\colon B \hookrightarrow \mathcal{P}(\mathrm{Hom}(B,2)), \quad b \mapsto \{x\colon x(b)=1\}$$

is injective.

Moreover, the topology τ on $\mathrm{spec}(B) := \mathrm{Hom}(B,2)$ generated by the image of $\widehat{(-)}$ is Boolean, and $B \cong \{$ clopens of $\mathrm{spec}(B)$ $\}$.

(A Boolean space is a compact space in which any distinct points are separable by a clopen.)

Example

For any set V, $\mathrm{spec}(\mathbb{F}_{\mathrm{BA}}(V))$ is the Cantor space 2^V.

Morphisms

Every homomorphism of Boolean algebras arises from a unique continuous function of spaces:

$$\begin{array}{c} B \\ \uparrow h \\ A \end{array} \qquad\Bigg\vert\qquad \begin{array}{c} \operatorname{spec} B \\ \downarrow h_* \\ \operatorname{spec} A \end{array}$$

We get an equivalence functor $\operatorname{spec}\colon \mathsf{BA}^{\mathrm{op}} \to \mathsf{BoolSp}$.

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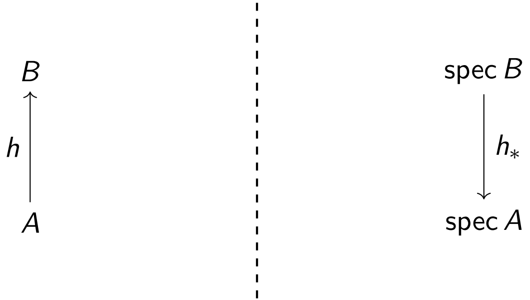

We get an equivalence functor spec: $\mathsf{BA}^{\mathrm{op}} \to \mathsf{BoolSp}$.

This is how we find unifiers: A unifier is a Boolean algebra homomorphism, so it must arise from a continuous function!

Boolean unification via duality

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$$(x_1 \wedge \neg x_2) \vee (x_2 \wedge x_3) \stackrel{?}{=} \top$$

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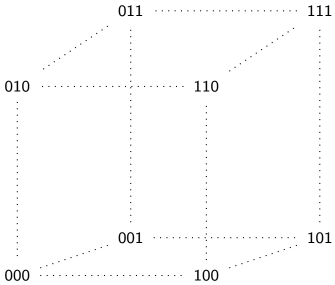
$$\mathbb{B}(x_1, x_2, x_3)$$



Boolean unification via duality

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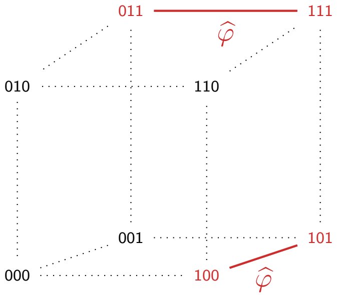

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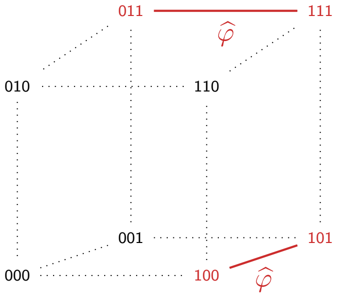

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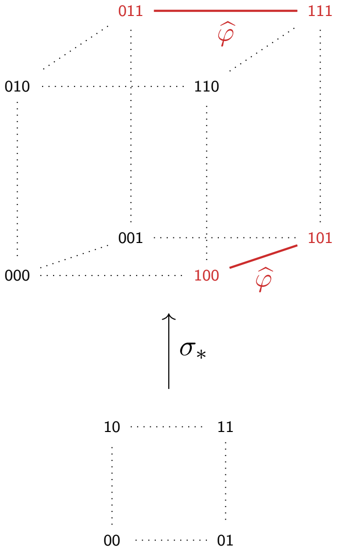

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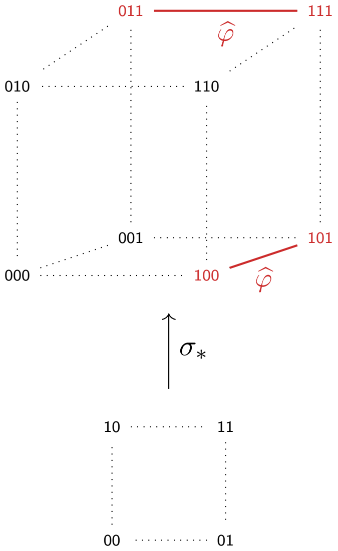

$$\sigma(\varphi) = \top \quad \Longleftrightarrow \quad \mathrm{im}(\sigma_*) \subseteq \widehat{\varphi}$$

Stone duality for (free) pointed dMA's

A pointed dMA is a tuple $(B, \mathbf{X}, c)$, where $\mathbf{X} \colon B \to B$ is a Boolean homomorphism and $c \in B$.

The dual category consists of tuples (X, f, K) where X is a Boolean space, $f \colon X \to X$ is continuous, and $K \subseteq X$ is clopen.

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Proposition

Let V a set of variables, $\mathbb{F}(V)$ the free pointed dMA on V. Then

$$\operatorname{spec} \mathbb{F}(V) = (\Sigma^\omega, s, \widehat{c}), \quad \text{where}$$

- $\Sigma := 2^{V \cup \{c\}}$, the local valuations over variables and constant;
- $s \colon \Sigma^\omega \to \Sigma^\omega$, the shift map, sends $(u_i)_{i=0}^\infty$ to $(u_i)_{i=1}^\infty$,
- $\widehat{c} = \{u \in \Sigma^\omega \mid u_0(c) = 1\}$.

Unifiers via duality

Let φ be a formula in variables V, WLOG of depth ≤ 1.

There are bijections between the following sets:

- $\equiv$-classes of ground unifiers of φ of depth $\leq d$;
- algebra homomorphisms $\sigma\colon \mathbb{F}(V) \to \mathbb{F}(\emptyset)$ such that each $\sigma(x)$ is of depth $\leq d$ and $\sigma(\varphi) = \top$;
- continuous shift-invariant maps $h\colon 2^\omega \to \Sigma^\omega$ with modulus of continuity d, and $\mathrm{im}(h) \subseteq \widehat{\varphi}$;
- de Bruijn graph homomorphisms $B_d(2) \to G(\varphi)$.

Admitting homomorphisms from de Bruijn graphs

Theorem (v.G., Marti, Sweering 2024)

For any finite graph $G = (V_G, E_G)$, the following are equivalent:

1. *there exist $d \geq 1$ and a surjective homomorphism $B_d(2) \to G$;*
2. *the graph G is cycle-connected and power-connected.*

Condition (2) can be verified in time $\mathcal{O}(\exp(|V_G| + |E_G|))$.

Corollary

Unifiability for pointed dMA's is decidable in 2-exponential time.

Take-aways

- De Bruijn graphs can open doors.
- Unification problems are hard to solve ad hoc.
- Stone duality can give a principled approach for solving them.

Take-aways

- De Bruijn graphs can open doors.
- Unification problems are hard to solve ad hoc.
- Stone duality can give a principled approach for solving them.

Thank you.

Further references

- van Gool & Marti, Modal unification step by step (2023).
- Gehrke & van Gool, Topological Duality for Distributive Lattices: Theory and Applications (2024).
- Baader & Ghilardi, Unification in Modal and Description Logics (2011).

Credits:

- Digicode picture: Wikipedia, user D4m1en, CC BY-SA 3.0.
- De Bruijn sequence ($k = 12$, $n = 4$) generated using code by Joe Sawada.

Appendix

Power-connectedness: intuition

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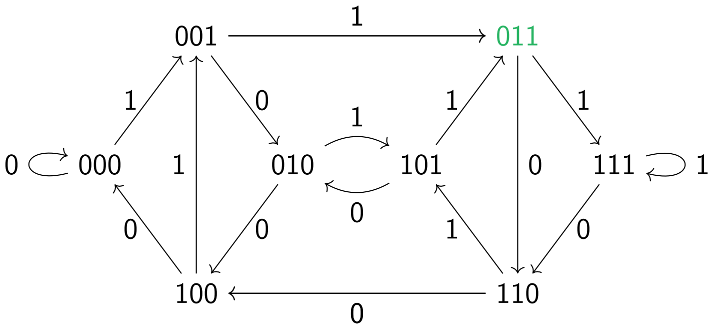

Letters read: 011

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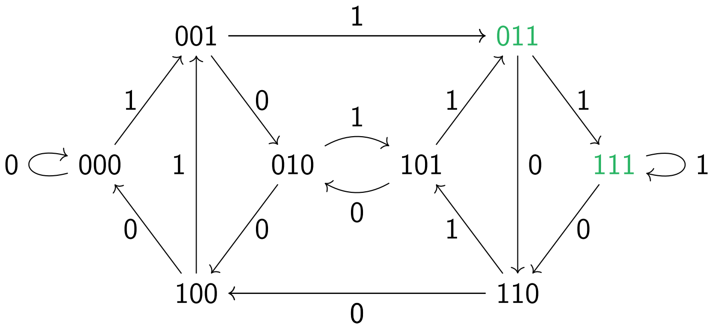

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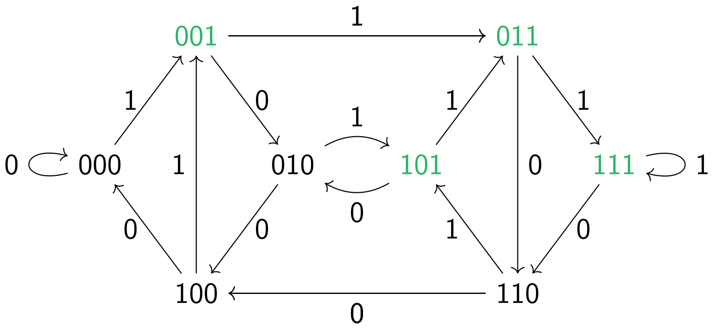

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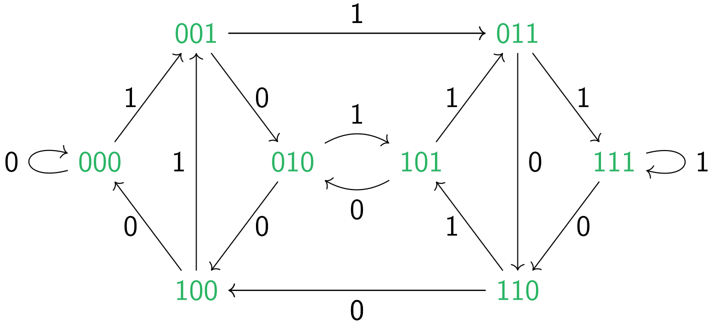

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Power-connectedness defined

Let $H = (V_H, E_H)$ be a Σ-graph.

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- G is power-connected if, in its power graph, the node V_G is in the closure of the set of nodes $\{\{u\} \ : \ u \in V_G\}$.

Cycle-connectedness: intuition

A *second* very strong connectedness property of de Bruijn graphs:

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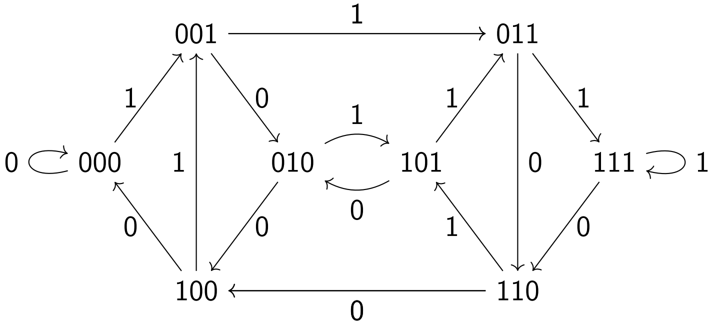

A cycle labeled 1100101

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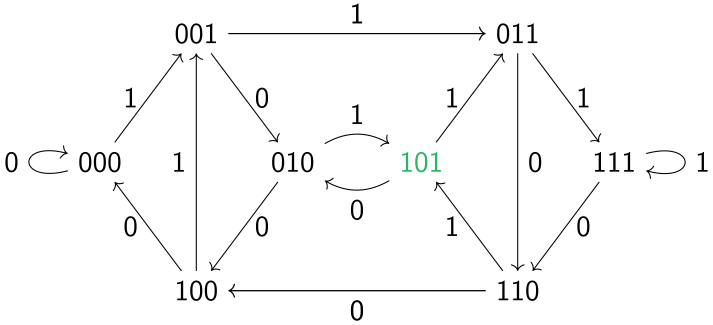

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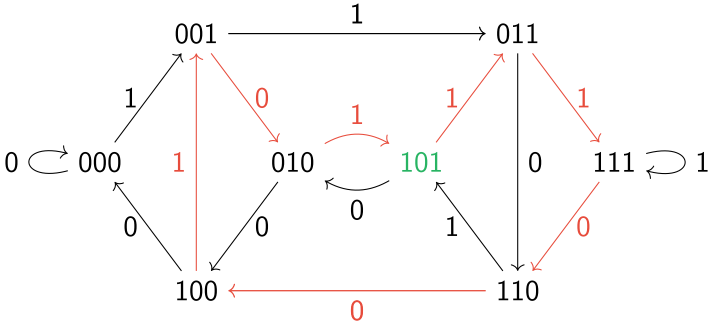

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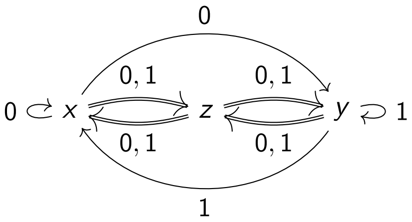
- A w-cycle $u \stackrel{w}{\to} u$ is reachable if there exists $k \geq 1$ such that for every node v, there is a path $v \stackrel{w^k}{\to} u$.
- G is cycle-connected if, for every $w \in \Sigma^+$, there exists a reachable w-cycle.

Answer for $d = 2$

◂ Back

The following 'hamburger' graph is not an image of B_2:



- In B_2, we have $00 \xrightarrow{1} 01 \xrightarrow{1} 11$; a homomorphism must map this to $x \xrightarrow{1} z \xrightarrow{1} y$.
- Similarly, $11 \xrightarrow{0} 10 \xrightarrow{0} 00$ must go to $y \xrightarrow{0} z \xrightarrow{0} x$.
- But now the edge $10 \xrightarrow{1} 01$ is not preserved: $z \overset{1}{\not\to} z$.

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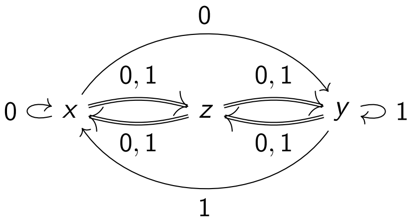

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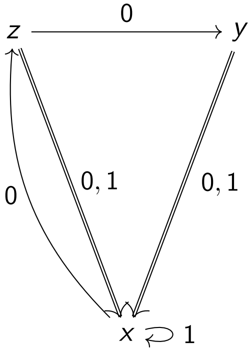
Another example

Back

Why can the following 'cone of fries' graph not admit a homomorphism from any B_d?



(This is a small example of a graph that is 'power-connected' but not 'cycle-connected', witnessing the independence of the two conditions in our characterization of images of de Bruijn graphs.)

Explanation

◂ Back

Suppose given an oracle for the surjective version. We solve the non-surjective version as follows.

For every subgraph G' of G, use the oracle for the surjective version to decide if a surjective homomorphism onto G' exists. If so, return this homomorphism, now viewed as a homomorphism to G. If the oracle for the surjective version always answers 'impossible', return impossible.

This is correct, because if some homomorphism h to G exists, then it will be surjective onto the subgraph $\mathrm{im}(h)$.

Explanation

◂ Back

Suppose φ were a formula such that $\varphi \leftrightarrow \neg \mathbf{X}\varphi$ is valid.

Consider the transition system with a single state, which is its own successor, and any valuation of the variables.

If φ evaluates to true in this transition system, then $\neg \mathbf{X}\varphi$ evaluates to true as well, by assumption on φ.

But then, by definition of the valuation, $\mathbf{X}\varphi$ should evaluate to false, so φ should evaluate to false.

The other case follows by symmetry.