Pointlike sets for varieties determined by groups

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Regular languages and profinite semigroups

Separation problems and pointlike sets

New result

Proof techniques

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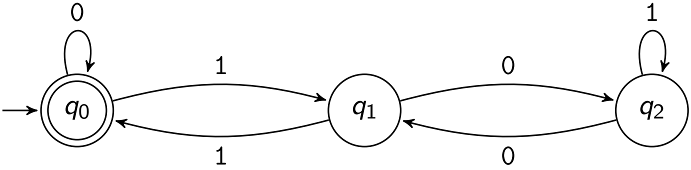

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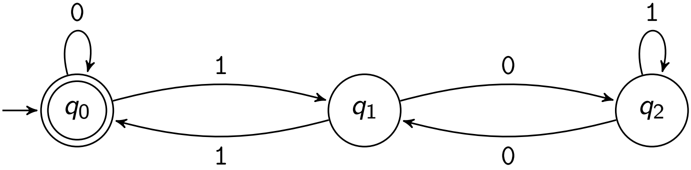

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- Solution 2: a regular expression

$$R := 0^*(1(01^*0)^*1)^*0^*$$

Answer yes iff w matches the expression R.

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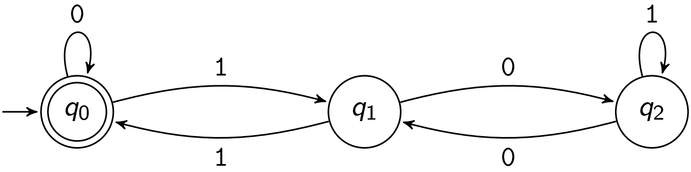

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- Solution 3: a homomorphism $\phi\colon \{0,1\}^+ \to S_3$

$$0 \mapsto (1\,2), \quad 1 \mapsto (0\,1).$$

Answer yes iff the permutation $\phi(w)$ fixes 0.

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Moreover, there is a computable 'minimal' such semigroup, called the syntactic semigroup of L.

Star-free languages

- A meta-problem: given a regular expression R, determine whether or not there is a star-free regular expression R' which describes the same language: $L(R) = L(R')$.

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- A semigroup as in (3) is called aperiodic.

Varieties

- A class of finite semigroups $\mathbf{V}$ is a variety if it is closed under finite products, homomorphic images, and subsemigroups.
- If $\mathbf{V}$ is a variety and Σ is a finite alphabet, $\mathcal{V}(\Sigma)$ denotes the set of languages $L \subseteq \Sigma^+$ with syntactic semigroup in $\mathbf{V}$.
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- The free profinite semigroup, $\widehat{\Sigma^+}$, maps onto the free pro-$\mathbf{V}$ semigroup with a map $\pi_{\mathbf{V}} \colon \widehat{\Sigma^+} \twoheadrightarrow \widehat{F}_{\mathbf{V}}(\Sigma)$.

Generalizing the membership problem

- An interpolation problem: given regular expressions R_1, R_2, determine whether there exists a star-free expression R such that $L(R_1) \subseteq L(R) \subseteq L(R_2)$.

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- The proof (Henckell 1988) translates the problem to a combinatorial question about a finite semigroup, namely, to compute its aperiodic-pointlike sets.

Pointlike sets

Proposition

Let $\mathbf{V}$ *be a variety of finite semigroups, and let* S *be a finite semigroup. For any subset* X *of* S*, the following are equivalent:*

1. *there exist profinite words* $x_1, \ldots, x_n \in \widehat{S^+}$ *such that* $X = \{[x_1]_S, \ldots, [x_n]_S\}$ *and* $\pi_{\mathbf{V}}(x_1) = \cdots = \pi_{\mathbf{V}}(x_n)$*;*

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Example. Any subgroup G of a finite semigroup S is $\mathbf{A}$-pointlike.

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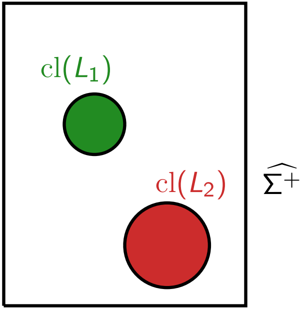

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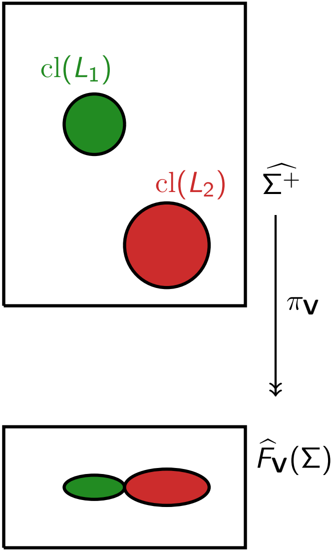

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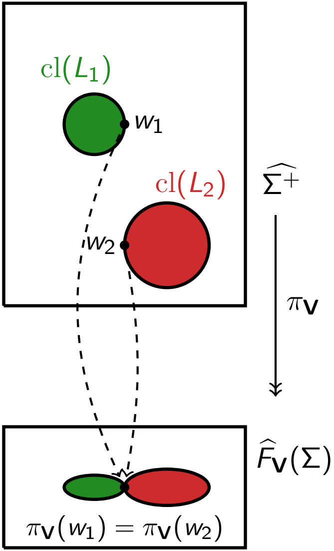

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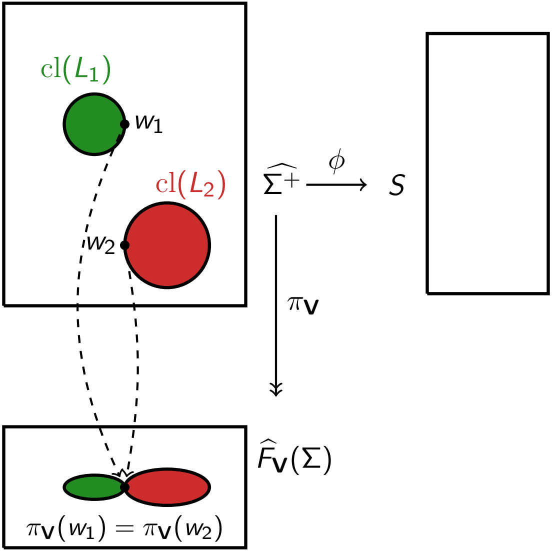

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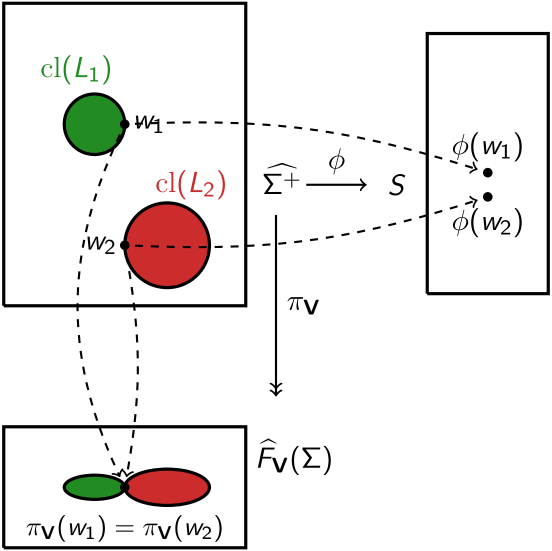

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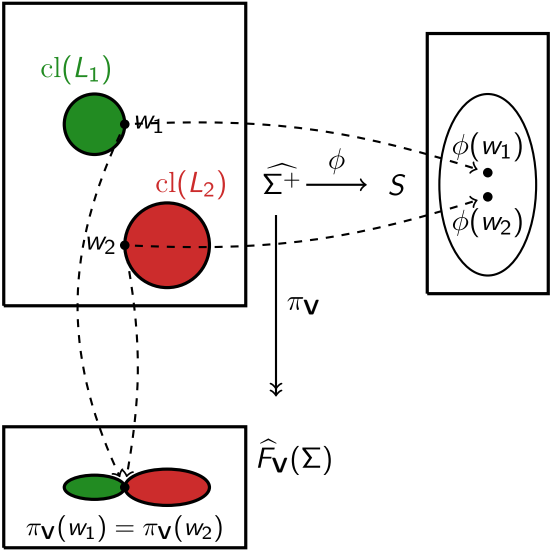

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- Thus, the separation problem for star-free languages is decidable.

Benjamin Steinberg

January 16 ·

Hard to believe I'm still working on pointlike sets after 20 years but...

ARXIV.ORG

[1801.04638] Pointlike sets for varieties determined by groups

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Joint work with Benjamin Steinberg (CCNY).

Generalizing aperiodic semigroups

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- For a variety of finite groups $\mathbf{H}$, define

$$\overline{\mathbf{H}} := \{S \text{ finite semigroup} : \text{all subgroups of } S \text{ are in } \mathbf{H}\}.$$

Theorem (G. & Steinberg 2018)

For any variety of finite groups $\mathbf{H}$ *with decidable membership, the* $\overline{\mathbf{H}}$*-pointlike sets are computable, and thus, in particular, the separation problem for* $\overline{\mathbf{H}}$*-languages is decidable.*

Corollaries

- $\mathbf{H} = \mathbf{1}$, the variety containing only the trivial group.
  $\Rightarrow$ Aperiodic-pointlikes computable (Henckell 1988)

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- $\mathbf{H} = \mathbf{G}_{\mathrm{sol}}$, the variety of solvable groups. Semigroups in $\overline{\mathbf{G}_{\mathrm{sol}}}$ are called solvable semigroups.
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Computing pointlikes

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Let $\mathbf{V}$ *be a variety of finite semigroups.*

The set of $\mathbf{V}$*-pointlikes,* $\mathrm{PL}_{\mathbf{V}}(S)$*, of a finite semigroup* S*, is:*

- *a subsemigroup of* $\mathcal{P}(S)$*:* $X \cdot Y$ *is* $\mathbf{V}$*-pointlike if* X *and* Y *are,*
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Moreover, $\mathrm{PL}_{\mathbf{V}}$ *is a submonad of* $\mathcal{P}$*:*

- *singletons are* $\mathbf{V}$*-pointlike,*
- *the union* $\bigcup \mathcal{X}$ *of any* $\mathbf{V}$*-pointlike subset* $\mathcal{X}$ *of the semigroup* $\mathrm{PL}_{\mathbf{V}}(S)$ *is* $\mathbf{V}$*-pointlike.*

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Difficult direction: every pointlike set is in the saturation.

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- Ingredients for building $S^{\mathbf{H}}$ and ρ:

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Completeness of the algorithm

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L-chains of $\mathbf{H}$-elements

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- Call $s \in S$ an $\mathbf{H}$-element if Γ_s lies in $\mathbf{H}$.

A blow-up operator on S

Proposition

There exists an idempotent operation $b\colon S \to S$ which fixes exactly the $\mathbf{H}$-elements, is $\leq_L$-contracting and $\subseteq$-expanding, i.e., for all $s \in S$, $b(s) \leq_L s$ and $s \subseteq b(s)$.

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- Composing b_0 sufficiently often with itself yields an idempotent operation b. $\square$

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  1. multiply every item in the chain $\mathbf{q}$ by t and add $\{t\}$ in front,
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Theorem

The semigroup $S^{\mathbf{H}}$ is a quotient of a subsemigroup of an infinite wreath product acting on S^*, which lies in $\overline{\mathbf{H}}$.

Final remarks

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- Computability of $\overline{\mathbf{H}}$-pointlikes implies decidability of membership for more involved varieties built from $\overline{\mathbf{H}}$; one may studies these algorithms for specific choices of $\mathbf{H}$.
- If the variety $\mathbf{H}$ is defined by profinite identities (e.g., trivial, abelian, solvable), these can be used to obtain a faster algorithm for computing $\overline{\mathbf{H}}$-pointlikes than the 'generic' saturation algorithm via $\mathbf{H}$-kernels.

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- A detailed study of the complexity and possible implementations for concrete $\mathbf{H}$ are future work.

Pointlike sets for varieties determined by groups

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Regular expressions for solvable semigroups

Theorem

A language is recognizable by a solvable semigroup iff it can be described by an AC-regular expression, i.e., an expression built up from Σ^*, Boolean operations, and, for any AC-regular expressions R, S, $a \in \Sigma$, prime p and $0 \le q < p$, the expressions RaS, and $(RaS)^{q \bmod p}$, which describes the language:

$$\{w \in \Sigma^* : \#\{(u,v) \in R \times S : w = uav\} = q \bmod p\}.$$

Example

Let $\Sigma = \{0, 1\}$ and $R = \Sigma^* \setminus (\Sigma^* 1 \Sigma^*)$, so that $L(R) = 0^*$.
The AC-expression $(R1R)^{q \bmod p}$ describes the language of words containing $q \mod p$ occurrences of 1.