Proaperiodic monoids via prime models

or: what do profinite words look like?

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joint work with Benjamin Steinberg, City College of New York

Structure meets Power

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Finite words and logic

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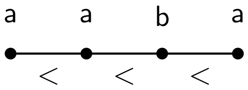

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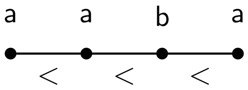

Every **monadic second order sentence** in signature

$$S_\Sigma := \{<\} \cup \{P_a \ : \ a \in \Sigma\}$$

describes a set of finite Σ-words.

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- L is recognizable by a finite automaton,
- L is saturated under a finite index monoid congruence on Σ^*, i.e., there exists a surjective homomorphism

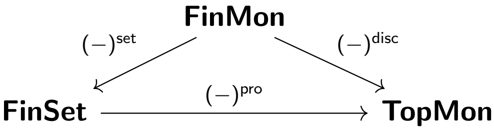
  $$h \colon \Sigma^* \twoheadrightarrow M,$$

  with M a finite monoid, such that, for some $P \subseteq M$,

  $$L = h^{-1}(P).$$

The free profinite monoid

The **free profinite monoid** over Σ is the, up to isomorphism unique, embedding of Σ into a topological monoid Σ^{pro} such that, for every finite monoid M and function $f\colon \Sigma \to M^{\mathsf{set}}$, there exists a unique continuous homomorphism $\overline{f}\colon \Sigma^{\mathsf{pro}} \to M^{\mathsf{disc}}$ that extends f.



$$\frac{f\colon \Sigma \to M^{\mathsf{set}}}{\overline{f}\colon \Sigma^{\mathsf{pro}} \to M^{\mathsf{disc}}}$$

Elements of Σ^{pro} are called **profinite words** over Σ.

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Aperiodicity is equivalent to the absence of non-trivial subgroups.

We get the monoid of **proaperiodic words** as the quotient

$$\Sigma^{\mathrm{ap}} := \Sigma^{\mathrm{pro}} / (x^{\omega} = x^{\omega} x).$$

Schützenberger 1965; McNaughton & Papert 1971; Reiterman 1982

Constructions of the free profinite monoid

Theorem. The topological space underlying the free profinite monoid Σ^{pro} can be constructed as:

- the limit in **TopMon** of a projective diagram of finite monoids,
- an ultrametric completion of Σ^*,
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Theorem. The multiplication on Σ^{pro} is dual to a residuation structure on the regular subsets of Σ.

Reiterman 1982; Gehrke, Grigorieff & Pin 2008

Constructions of the free profinite monoid

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Similarly, the equivalence

$$\text{aperiodic} \Leftrightarrow \text{FO-definable}$$

induces a homeomorphism

$$\Sigma^{\text{ap}} \cong \text{completions of the FO-theory of finite words.}$$

Steinberg & G. 2016; Linkhorn 2021

What do proaperiodic words look like? (I)

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Proposition (model theory). Every elementary equivalence class of pseudofinite Σ-words contains an ω-**saturated** model.

This can be used, e.g., to solve the word problem of $\langle \Sigma^{\mathsf{ap}}, \cdot, ()^{\omega} \rangle$.

Steinberg & G. 2016

Example (I)

The saturated model representing the proaperiodic word $(ab)^\omega$ is

$$ababab\ldots \quad (\ldots ababab\ldots)^{\mathbb{Q}} \quad \ldots ababab,$$

where the middle part has order type $\mathbb{Q} \times_{\mathrm{lex}} \mathbb{Z}$.

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A calculus: finite structures are saturated, substitution of saturated models into saturated models remains saturated.

There also exist proaperiodic words with uncountably many factors (these are a bit harder to draw).

Proaperiodic words and step points

An alternative way of "realizing" profinite words as structures:

Definition. For a proaperiodic word u, define a **point** of u as a pair $(u_1, u_2) \in (\Sigma^{\mathsf{ap}})^2$ such that $u_1 u_2 = u$.

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The set of points of u may be linearly ordered, in such a way that the following theorem holds:

Theorem. A proaperiodic word can be fully described by a labeled linear order of its **step points**, i.e., those points that have an immediate predecessor and successor, or none at all.

Almeida, A. Costa, J. Costa, Zeitoun 2017

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The step point structure fits well with our point of view.

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Recall that a model of a theory T is **prime** if it embeds elementarily into every model of T.

Theorem (model theory + some work). Every elementary class of pseudofinite words contains a **prime** model, which is isomorphic to the step point structure (up to an off-by-one error), and multiplication of step point structures is just concatenation of prime models.

Steinberg & G. 202x

Example (II)

The prime model representing the proaperiodic word $(ab)^\omega$ is

$$ababab\ldots \qquad\qquad \ldots ababab,$$

where the middle part has disappeared.

What do profinite words look like?

Some possible further directions:

- What happens for MSO on more than one letter?
- What happens for fragments of FO (in particular $B\Sigma_n$)?
- What happens for profinite structures other than words?
- What is the correct categorical point of view on all of this?