Theory and Practice of Uniform Interpolation

Sam van Gool

IRIF, Université Paris Cité

WoLLIC, Bern, 13 June 2024

Overview

Uniform interpolation

Practice

Theory

Overview

Uniform interpolation

Practice

Theory

Interpolation

Interpolation is the problem that asks, given a deduction

$$A \vdash B$$

to find C such that

$$A \vdash C \vdash B$$

and C only uses symbols that are in both A and B.

Interpolation

Interpolation is the problem that asks, given a deduction

$$A \vdash B$$

to find C such that

$$A \vdash C \vdash B$$

and C only uses symbols that are in both A and B.

- What are A, B, C? Which symbols? What is $\vdash$?

Interpolation

Interpolation is the problem that asks, given a deduction

$$A \vdash B$$

to find C such that

$$A \vdash C \vdash B$$

and C only uses symbols that are in both A and B.

- What are A, B, C? Which symbols? What is $\vdash$?

We will look at propositional logics, and take symbols to mean propositional variables.

The classical case

Suppose that

$$A(p, q) \vdash B(p, r)$$

for propositional formulas A and B.

The classical case

Suppose that

$$A(p, q) \vdash B(p, r)$$

for propositional formulas A and B.

If $\vdash$ is classical entailment, then the formula

$$C(p) := A(p, \bot) \vee A(p, \top)$$

is an interpolant:

$$A(p, q) \vdash C(p) \vdash B(p, r).$$

The classical case

Suppose that

$$A(p, q) \vdash B(p, r)$$

for propositional formulas A and B.

If $\vdash$ is classical entailment, then the formula

$$C(p) := A(p, \bot) \vee A(p, \top)$$

is an interpolant:

$$A(p, q) \vdash C(p) \vdash B(p, r).$$

So is

$$C'(p) := B(p, \bot) \wedge B(p, \top).$$

Uniform interpolants

Note that each of the interpolants

$$C(p) := A(p,\bot) \vee A(p,\top) \quad \text{and} \quad C'(p) := B(p,\bot) \wedge B(p,\top)$$

only depends on one of the formulas in the entailment $A \vdash B$.

Uniform interpolants

Note that each of the interpolants

$$C(p) := A(p, \bot) \vee A(p, \top) \quad \text{and} \quad C'(p) := B(p, \bot) \wedge B(p, \top)$$

only depends on one of the formulas in the entailment $A \vdash B$.

These uniform interpolants encode propositional quantifiers:

$$C(p) \equiv \exists q.\ A(p, q) \quad \text{and} \quad C'(p) \equiv \forall q.\ B(p, q)\ .$$

Uniform interpolants

Note that each of the interpolants

$$C(p) := A(p, \bot) \vee A(p, \top) \quad \text{and} \quad C'(p) := B(p, \bot) \wedge B(p, \top)$$

only depends on one of the formulas in the entailment $A \vdash B$.

These uniform interpolants encode propositional quantifiers:

$$C(p) \equiv \exists q.\ A(p, q) \quad \text{and} \quad C'(p) \equiv \forall q.\ B(p, q)\ .$$

The simple encoding works because classical logic is locally finite: If we fix a finite set of variables, then there are only finitely many equivalence classes of formulas with variables from this set.

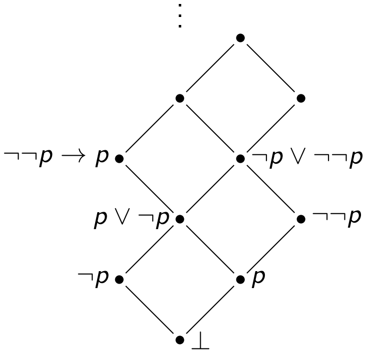
The intuitionistic case

Intuitionistic Propositional Logic is not locally finite.

Even for just 1 variable, we have infinitely many non-equivalent formulas:

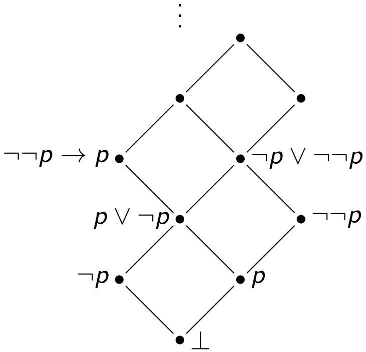
The intuitionistic case

Intuitionistic Propositional Logic is not locally finite.
Even for just 1 variable, we have infinitely many non-equivalent formulas:



The intuitionistic case

Intuitionistic Propositional Logic is not locally finite.
Even for just 1 variable, we have infinitely many non-equivalent formulas:



Still, we have:

Theorem (Pitts 1992)

There exists a computable encoding of propositional quantifiers in intuitionistic propositional logic.

Detailed statement of Pitts' Theorem

For every propositional formula $\varphi(\bar{p}, q)$, one can compute q-free formulas

$$E_q(\varphi) \quad \text{and} \quad A_q(\varphi),$$

with variables in $\bar{p}$,

Detailed statement of Pitts' Theorem

For every propositional formula $\varphi(\bar{p}, q)$, one can compute q-free formulas

$$E_q(\varphi) \quad \text{and} \quad A_q(\varphi),$$

with variables in $\bar{p}$, such that, for any q-free formula ψ,

$$\text{if } \varphi \vdash \psi \text{ then } \varphi \vdash E_q\varphi \vdash \psi \ ,$$

Detailed statement of Pitts' Theorem

For every propositional formula $\varphi(\bar{p}, q)$, one can compute q-free formulas

$$E_q(\varphi) \quad \text{and} \quad A_q(\varphi),$$

with variables in $\bar{p}$, such that, for any q-free formula ψ,

$$\text{if } \varphi \vdash \psi \text{ then } \varphi \vdash E_q\varphi \vdash \psi \ ,$$

and

$$\text{if } \psi \vdash \varphi \text{ then } \psi \vdash A_q\varphi \vdash \varphi \ ,$$

where $\varphi \vdash \psi$ means intuitionistic entailment.

Aside: Why Pitts proved his theorem

"Some ten or so years ago I tried to prove the negation of [the theorem] in connection with (...) the question of whether any Heyting algebra can appear as the algebra of truth-values of an elementary topos. I established that the free Heyting algebra on a countable infinity of generators does not so appear provided [the theorem] does not hold. It seemed likely to me (and to others to whom I posed the question) that a [formula] φ could be found for which $A_p\varphi$ does not exist (although I could not find one!), thus settling the original question about toposes and Heyting algebras in the negative. That [the theorem] is true is quite a surprise to me. (...) It remains an open question whether every Heyting algebra can be the Lindenbaum algebra of a theory in intuitionistic higher order logic."

(Pitts, 1992)

Intuitionistic propositional quantifiers

In IPC, the simple computation of E_q from the classical setting no longer works.

Intuitionistic propositional quantifiers

In IPC, the simple computation of E_q from the classical setting no longer works. For example, when

$$\varphi = (\neg p \to q) \wedge (q \to r)$$

we have

$$\varphi[\bot/q] \equiv \neg\neg p, \quad \varphi[\top/q] \equiv r$$

but

$$\varphi \nvdash \neg\neg p \vee r \ .$$

Intuitionistic propositional quantifiers

In IPC, the simple computation of E_q from the classical setting no longer works. For example, when

$$\varphi = (\neg p \to q) \wedge (q \to r)$$

we have

$$\varphi[\bot/q] \equiv \neg\neg p, \quad \varphi[\top/q] \equiv r$$

but

$$\varphi \nvdash \neg\neg p \vee r \ .$$

In this example, it turns out that $E_q(\varphi)$ can be computed as

$$\neg p \to r \ ,$$

which is equivalent to $\varphi[\neg p/q]$.

A finite basis for interpolants

Given a formula $\varphi(\bar{p}, q)$, we have

$$\varphi(\bar{p}, q) \vdash \bigwedge \{\psi(\bar{p}) \mid \varphi \vdash \psi\}.$$

A finite basis for interpolants

Given a formula $\varphi(\bar{p}, q)$, we have

$$\varphi(\bar{p}, q) \vdash \bigwedge \{\psi(\bar{p}) \mid \varphi \vdash \psi\}.$$

The expression on the right is q-free, but not a formula.

A finite basis for interpolants

Given a formula $\varphi(\bar{p}, q)$, we have

$$\varphi(\bar{p}, q) \vdash \bigwedge\{\psi(\bar{p}) \ \mid \ \varphi \vdash \psi\}.$$

The expression on the right is q-free, but not a formula.

The idea is to replace it by

$$E_q(\varphi) \stackrel{\text{def}}{=} \bigwedge \mathcal{E}_q(\varphi)$$

where $\mathcal{E}_q(\varphi)$ is a finite basis for the set of consequences of φ.

A finite basis for interpolants

Given a formula $\varphi(\bar{p}, q)$, we have

$$\varphi(\bar{p}, q) \vdash \bigwedge \{\psi(\bar{p}) \mid \varphi \vdash \psi\}.$$

The expression on the right is q-free, but not a formula.

The idea is to replace it by

$$E_q(\varphi) \stackrel{\text{def}}{=} \bigwedge \mathcal{E}_q(\varphi)$$

where $\mathcal{E}_q(\varphi)$ is a finite basis for the set of consequences of φ.
The computation of $A_q(\varphi)$ is similar, using a disjunction of $\mathcal{A}_q(\varphi)$.
Pitts' definition recurses on the shape of the formula A, using already computed sets $\mathcal{E}_q(\varphi')$ and $\mathcal{A}_q(\varphi')$ for smaller formulas φ'.

Computing intuitionistic propositional quantifiers

Pitts constructs quantifiers, and proves correctness, by induction on proofs of $A \vdash B$.

The idea is that $E_p(A)$ represents 'all possible consequences of A in a finite terminating proof search' (Iemhoff 2019, v.d.Giessen 2023).

Computing intuitionistic propositional quantifiers

Pitts constructs quantifiers, and proves correctness, by induction on proofs of $A \vdash B$.

The idea is that $E_p(A)$ represents 'all possible consequences of A in a finite terminating proof search' (Iemhoff 2019, v.d.Giessen 2023).

- What proof calculus to use?

A terminating sequent calculus

Gentzen calculus **LJ** has contraction, and the rule:

$$\frac{\Gamma, \varphi_1 \to \varphi_2 \vdash \varphi_1 \qquad \Gamma, \varphi_2 \vdash \psi}{\Gamma, \varphi_1 \to \varphi_2 \vdash \psi}$$

which make proof search not obviously terminating.

A terminating sequent calculus

Gentzen calculus **LJ** has contraction, and the rule:

$$\frac{\Gamma, \varphi_1 \to \varphi_2 \vdash \varphi_1 \qquad \Gamma, \varphi_2 \vdash \psi}{\Gamma, \varphi_1 \to \varphi_2 \vdash \psi}$$

which make proof search not obviously terminating.

Classical solution: **G4ip** uses multisets as sequents, and replaces the $\to$-left rule by a finer case analysis on φ_1.

G4ip

Replace $\rightarrow$-left rule by the following four rules:

G4ip

Replace $\rightarrow$-left rule by the following four rules:

$$\frac{\mathcal{F}, p, A \vdash C}{\mathcal{F}, p, p \rightarrow A \vdash C}$$

G4ip

Replace $\rightarrow$-left rule by the following four rules:

$$\frac{\mathcal{F}, p, A \vdash C}{\mathcal{F}, p, p \rightarrow A \vdash C} \qquad \frac{\mathcal{F}, A_1 \rightarrow (A_2 \rightarrow B) \vdash C}{\mathcal{F}, (A_1 \wedge A_2) \rightarrow B \vdash C}$$

G4ip

Replace $\rightarrow$-left rule by the following four rules:

$$\frac{\mathcal{F}, p, A \vdash C}{\mathcal{F}, p, p \rightarrow A \vdash C} \qquad \frac{\mathcal{F}, A_1 \rightarrow (A_2 \rightarrow B) \vdash C}{\mathcal{F}, (A_1 \wedge A_2) \rightarrow B \vdash C}$$

$$\frac{\mathcal{F}, A_1 \rightarrow B, A_2 \rightarrow B \vdash C}{\mathcal{F}, (A_1 \vee A_2) \rightarrow B \vdash C}$$

G4ip

Replace $\rightarrow$-left rule by the following four rules:

$$\frac{\mathcal{F}, p, A \vdash C}{\mathcal{F}, p, p \rightarrow A \vdash C} \qquad \frac{\mathcal{F}, A_1 \rightarrow (A_2 \rightarrow B) \vdash C}{\mathcal{F}, (A_1 \wedge A_2) \rightarrow B \vdash C}$$

$$\frac{\mathcal{F}, A_1 \rightarrow B, A_2 \rightarrow B \vdash C}{\mathcal{F}, (A_1 \vee A_2) \rightarrow B \vdash C} \quad \frac{\mathcal{F}, A_2 \rightarrow B \vdash A_1 \rightarrow A_2 \qquad \mathcal{F}, B \vdash C}{\mathcal{F}, (A_1 \rightarrow A_2) \rightarrow B \vdash C}$$

G4ip

Replace $\rightarrow$-left rule by the following four rules:

$$\frac{\mathcal{F}, p, A \vdash C}{\mathcal{F}, p, p \rightarrow A \vdash C} \qquad \frac{\mathcal{F}, A_1 \rightarrow (A_2 \rightarrow B) \vdash C}{\mathcal{F}, (A_1 \wedge A_2) \rightarrow B \vdash C}$$

$$\frac{\mathcal{F}, A_1 \rightarrow B, A_2 \rightarrow B \vdash C}{\mathcal{F}, (A_1 \vee A_2) \rightarrow B \vdash C} \quad \frac{\mathcal{F}, A_2 \rightarrow B \vdash A_1 \rightarrow A_2 \qquad \mathcal{F}, B \vdash C}{\mathcal{F}, (A_1 \rightarrow A_2) \rightarrow B \vdash C}$$

G4ip

Replace $\rightarrow$-left rule by the following four rules:

$$\frac{\mathcal{F}, p, A \vdash C}{\mathcal{F}, p, p \rightarrow A \vdash C} \qquad \frac{\mathcal{F}, A_1 \rightarrow (A_2 \rightarrow B) \vdash C}{\mathcal{F}, (A_1 \wedge A_2) \rightarrow B \vdash C}$$

$$\frac{\mathcal{F}, A_1 \rightarrow B, A_2 \rightarrow B \vdash C}{\mathcal{F}, (A_1 \vee A_2) \rightarrow B \vdash C} \quad \frac{\mathcal{F}, A_2 \rightarrow B \vdash A_1 \rightarrow A_2 \qquad \mathcal{F}, B \vdash C}{\mathcal{F}, (A_1 \rightarrow A_2) \rightarrow B \vdash C}$$

Theorem

The sequent calculus **G4ip** *is terminating, sound and complete for intuitionistic propositional logic.*

G4ip

Replace $\to$-left rule by the following four rules:

$$\frac{\mathcal{F}, p, A \vdash C}{\mathcal{F}, p, p \to A \vdash C} \qquad \frac{\mathcal{F}, A_1 \to (A_2 \to B) \vdash C}{\mathcal{F}, (A_1 \wedge A_2) \to B \vdash C}$$

$$\frac{\mathcal{F}, A_1 \to B, A_2 \to B \vdash C}{\mathcal{F}, (A_1 \vee A_2) \to B \vdash C} \qquad \frac{\mathcal{F}, A_2 \to B \vdash A_1 \to A_2 \qquad \mathcal{F}, B \vdash C}{\mathcal{F}, (A_1 \to A_2) \to B \vdash C}$$

Theorem

The sequent calculus **G4ip** *is terminating, sound and complete for intuitionistic propositional logic.*

(Originally discovered by Vorob'ev 1952. Hudelmaier 1988 rediscovered it. Dyckhoff 1992 popularized it as '**LJT**'. Troelsta & Schwichtenberg 1996 introduced the name '**G4ip**'.)

A glimpse at Pitts' table

	Δ matches:	$\mathcal{E}(\Delta)$ contains:
E_1	$\Delta' \bullet q$	$E(\Delta') \wedge q$
E_4	$\Delta' \bullet (q \to \delta)$	$q \to E(\Delta' \bullet \delta)$
E_5	$\Delta'' \bullet p \bullet (p \to \delta)$	$E(\Delta'' \bullet p \bullet \delta)$
E_6	$\Delta' \bullet (\delta_1 \wedge \delta_2) \to \delta_3$	$E(\Delta' \bullet (\delta_1 \to (\delta_2 \to \delta_3)))$
E_8	$\Delta' \bullet ((\delta_1 \to \delta_2) \to \delta_3)$	$(E(\Delta' \bullet (\delta_2 \to \delta_3)) \to A(\Delta' \bullet (\delta_2 \to \delta_3), \delta_1 \to \delta_2)) \to E(\Delta' \bullet \delta_3)$
	Δ, ϕ matches:	$\mathcal{A}(\Delta, \phi)$ contains:
A_3	$\Delta' \bullet \delta_1 \vee \delta_2, \phi$	$(E(\Delta' \bullet \delta_1) \to A(\Delta' \bullet \delta_1, \phi)) \wedge (E(\Delta' \bullet \delta_2) \to A(\Delta' \bullet \delta_2, \phi))$
A_7	$\Delta' \bullet (\delta_1 \vee \delta_2) \to \delta_3, \phi$	$A(\Delta' \bullet (\delta_1 \to \delta_3) \bullet (\delta_2 \to \delta_3), \phi)$
A_8	$\Delta' \bullet ((\delta_1 \to \delta_2) \to \delta_3), \phi$	$(E(\Delta' \bullet (\delta_2 \to \delta_3)) \to A(\Delta' \bullet (\delta_2 \to \delta_3), (\delta_1 \to \delta_2))) \wedge A(\Delta' \bullet \delta_3, \phi)$
A_{11}	$\Delta, \phi_1 \wedge \phi_2$	$A(\Delta, \phi_1) \wedge A(\Delta, \phi_2)$
A_{12}	$\Delta, \phi_1 \vee \phi_2$	$A(\Delta, \phi_1) \vee A(\Delta, \phi_2)$
A_{13}	$\Delta, \phi_1 \to \phi_2$	$E(\Delta \bullet \phi_1, \phi_2) \to A(\Delta \bullet \phi_1, \phi_2)$

Table 1. Excerpt of Pitts' definitions of $\mathcal{E}(\Delta)$ and $\mathcal{A}(\Delta, \phi)$, with respect to a fixed variable p.

Overview

Uniform interpolation

Practice

Theory

Pitts verified

In joint work with H. Férée (CPP 2023), we formalized Pitts' construction and correctness proof in Coq, yielding a correct-by-construction program that computes E_p and A_p.

`https://hferee.github.io/UIML/`

Pitts verified

In joint work with H. Férée (CPP 2023), we formalized Pitts' construction and correctness proof in Coq, yielding a correct-by-construction program that computes E_p and A_p.

`https://hferee.github.io/UIML/`

- Intricate properties of the proof calculus play a big role.
- We obtain a usable program (with optimizations to be done).
- Recently, with Férée, v.d. Giessen and Shillito (IJCAR 2024): Extension of formalization to **K**, **GL**, and **iSL**.

Pitts verified

In joint work with H. Férée (CPP 2023), we formalized Pitts' construction and correctness proof in Coq, yielding a correct-by-construction program that computes E_p and A_p.

`https://hferee.github.io/UIML/`

- Intricate properties of the proof calculus play a big role.
- We obtain a usable program (with optimizations to be done).
- Recently, with Férée, v.d. Giessen and Shillito (IJCAR 2024): Extension of formalization to **K**, **GL**, and **iSL**.
  Open problems:
- How to make it (even) more modular?
- How to tackle difficult cases (**iGL**)?

Overview

Uniform interpolation

Practice

Theory

The algebraic approach

Intuitionistic propositional logic is algebraically interpreted by Heyting algebras: structures $(H, \vee, \wedge, \bot, \top, \to)$ satisfying the axioms of a bounded distributive lattice and, for all $a, b, c \in H$,

$$a \wedge b \leq c \iff a \leq b \to c \ .$$

The algebraic approach

Intuitionistic propositional logic is algebraically interpreted by Heyting algebras: structures $(H, \vee, \wedge, \bot, \top, \to)$ satisfying the axioms of a bounded distributive lattice and, for all $a, b, c \in H$,

$$a \wedge b \leq c \iff a \leq b \to c \ .$$

A Heyting category (aka logos) is a coherent category in which all change of base functors have upper and lower adjoints.

Pitts' Theorem, semantically

Pitts' theorem can be reformulated using Heyting algebras as:

Theorem (Pitts)

Any homomorphism between finitely generated free Heyting algebras has both an upper and a lower adjoint.

Pitts' Theorem, semantically

Pitts' theorem can be reformulated using Heyting algebras as:

Theorem (Pitts)

Any homomorphism between finitely generated free Heyting algebras has both an upper and a lower adjoint.

A further consequence of this is:

Theorem (Pitts; Ghilardi & Zawadowski)

The opposite of the category $\mathbf{HA}_{\mathrm{fp}}$ *of finitely presented Heyting algebras is a Heyting category.*

A proof via sheaves

S. Ghilardi and M. Zawadowski (1995) gave a new, semantic proof of Pitts' theorem.

A proof via sheaves

S. Ghilardi and M. Zawadowski (1995) gave a new, semantic proof of Pitts' theorem. They start from the observation that every finitely presented Heyting algebra H can be faithfully represented by a covariant presheaf

$$\Phi_H \colon \mathbf{HA}_{\mathrm{fin}} \longrightarrow \mathbf{Set}$$

defined as the restriction of $\mathrm{Hom}(H, -)$ to finite algebras.

A proof via sheaves

S. Ghilardi and M. Zawadowski (1995) gave a new, semantic proof of Pitts' theorem. They start from the observation that every finitely presented Heyting algebra H can be faithfully represented by a covariant presheaf

$$\Phi_H \colon \mathbf{HA}_{\mathrm{fin}} \longrightarrow \mathbf{Set}$$

defined as the restriction of $\mathrm{Hom}(H, -)$ to finite algebras.

G&Z notice that Φ_H can also be seen as a contravariant sheaf on the category $\mathbf{Pos}_{\mathrm{fin}}$ of finite posets, giving a functor

$$\Phi \colon \mathbf{HA}_{\mathrm{fp}} \longrightarrow \mathrm{Sh}(\mathbf{Pos}_{\mathrm{fin}}),$$

A proof via sheaves

S. Ghilardi and M. Zawadowski (1995) gave a new, semantic proof of Pitts' theorem. They start from the observation that every finitely presented Heyting algebra H can be faithfully represented by a covariant presheaf

$$\Phi_H \colon \mathbf{HA}_{\mathrm{fin}} \longrightarrow \mathbf{Set}$$

defined as the restriction of $\mathrm{Hom}(H, -)$ to finite algebras.

G&Z notice that Φ_H can also be seen as a contravariant sheaf on the category $\mathbf{Pos}_{\mathrm{fin}}$ of finite posets, giving a functor

$$\Phi \colon \mathbf{HA}_{\mathrm{fp}} \longrightarrow \mathrm{Sh}(\mathbf{Pos}_{\mathrm{fin}}),$$

and characterize the image of Φ via a combinatorial condition $(*)$.

A proof via sheaves

S. Ghilardi and M. Zawadowski (1995) gave a new, semantic proof of Pitts' theorem. They start from the observation that every finitely presented Heyting algebra H can be faithfully represented by a covariant presheaf

$$\Phi_H \colon \mathbf{HA}_{\mathrm{fin}} \longrightarrow \mathbf{Set}$$

defined as the restriction of $\mathrm{Hom}(H, -)$ to finite algebras.
G&Z notice that Φ_H can also be seen as a contravariant sheaf on the category $\mathbf{Pos}_{\mathrm{fin}}$ of finite posets, giving a functor

$$\Phi \colon \mathbf{HA}_{\mathrm{fp}} \longrightarrow \mathrm{Sh}(\mathbf{Pos}_{\mathrm{fin}}),$$

and characterize the image of Φ via a combinatorial condition $(*)$.
They prove Pitts' Theorem by showing that the direct image $(\exists)$ and universal image $(\forall)$ operations on sheaves preserve $(*)$.

Quantifier elimination from uniform interpolation

Ghilardi and Zawadowski use Pitts' theorem to prove:

Theorem. The theory of Heyting algebras has a model completion.

Quantifier elimination from uniform interpolation

Ghilardi and Zawadowski use Pitts' theorem to prove:

Theorem. The theory of Heyting algebras has a model completion.

Here, a model completion of a first order theory is an extension with quantifier elimination and the same universal theory.

Quantifier elimination from uniform interpolation

Ghilardi and Zawadowski use Pitts' theorem to prove:

Theorem. The theory of Heyting algebras has a model completion.

Here, a model completion of a first order theory is an extension with quantifier elimination and the same universal theory.

One may identify the algebraic conditions needed for this, giving a modular approach to model completions (Ghilardi & Zawadowski 2002; vG., Tsinakis, Metcalfe 2017; Metcalfe & Reggio 2023).

Quantifier elimination from uniform interpolation

Ghilardi and Zawadowski use Pitts' theorem to prove:

Theorem. The theory of Heyting algebras has a model completion.

Here, a model completion of a first order theory is an extension with quantifier elimination and the same universal theory.

One may identify the algebraic conditions needed for this, giving a modular approach to model completions (Ghilardi & Zawadowski 2002; vG., Tsinakis, Metcalfe 2017; Metcalfe & Reggio 2023).

Further direction: Model completions for other varieties of logic-related algebras (LTL, CTL, . . . , see Ghilardi & vG. 2016–. . .)

Pitts via duality

A re-interpretation of the G&Z sheaf-theoretic proof.

Pitts via duality

A re-interpretation of the G&Z sheaf-theoretic proof.

Any bounded distributive lattice H can be described as a lattice of compact-open subsets of a topological space X, based on the set

$$\mathbf{DL}(H, 2)$$

of homomorphisms to the two-element lattice (Stone 1937).

Pitts via duality

A re-interpretation of the G&Z sheaf-theoretic proof.

Any bounded distributive lattice H can be described as a lattice of compact-open subsets of a topological space X, based on the set

$$\mathbf{DL}(H, 2)$$

of homomorphisms to the two-element lattice (Stone 1937).

Esakia (1974) derived from this a dual equivalence between Heyting algebras and certain *ordered compact spaces*, now called Esakia spaces. The finite part is Kripke semantics.

Advertising break

Now in print!

20% discount flyer available by e-mail from the authors (vangool@irif.fr)

M. Gehrke & SvG: *Topological Duality for Distributive Lattices: Theory and Applications*, Cambridge University Press, 369pp (2024).

Esakia spaces

An Esakia space is a compact ordered space that is totally order disconnected and such that $\uparrow U$ is open for every open set U.

Esakia spaces

An Esakia space is a compact ordered space that is totally order disconnected and such that $\uparrow U$ is open for every open set U.

The main Esakia space of interest here is the canonical model, $X(\bar{p})$, over a finite set of variables $\bar{p}$:

- points are prime theories in variables $\bar{p}$;
- order is inclusion of theories;
- topology is generated by $\widehat{\varphi} := \{x \in X(\bar{p}) \mid \varphi \in x\}$.

Esakia spaces

An Esakia space is a compact ordered space that is totally order disconnected and such that $\uparrow U$ is open for every open set U.

The main Esakia space of interest here is the canonical model, $X(\bar{p})$, over a finite set of variables $\bar{p}$:

- points are prime theories in variables $\bar{p}$;
- order is inclusion of theories;
- topology is generated by $\widehat{\varphi} := \{x \in X(\bar{p}) \mid \varphi \in x\}$.

A co-finitely presented Esakia space is one that is isomorphic to a clopen up-set of $X(\bar{p})$, for some finite $\bar{p}$.

An open mapping theorem

We give an open mapping theorem for Esakia spaces:

Theorem (vG. & Reggio 2018)

Every continuous monotone bounded map between co-finitely presented Esakia spaces is open.

An open mapping theorem

We give an open mapping theorem for Esakia spaces:

Theorem (vG. & Reggio 2018)

Every continuous monotone bounded map between co-finitely presented Esakia spaces is open.

By Esakia duality, this implies the algebraic Pitts' Theorem:

Corollary

Every homomorphism between finitely presented Heyting algebras has a lower and upper adjoint.

Definable bisimulation quantifiers

First main idea in all semantic proofs (see also Visser, 1996):

uniform interpolation $\leftrightarrow$ definability of bisimulation quantifiers.

Definable bisimulation quantifiers

First main idea in all semantic proofs (see also Visser, 1996):

uniform interpolation $\leftrightarrow$ definability of bisimulation quantifiers.

A $\bar{p}$-model is a poset $(X, \leq)$, with a function $v \colon \bar{p} \to \mathrm{Up}(X, \leq)$.
By induction, any formula φ gets a semantics $[\![\varphi]\!]_X \in \mathrm{Up}(X, \leq)$.

Definable bisimulation quantifiers

First main idea in all semantic proofs (see also Visser, 1996):

uniform interpolation $\leftrightarrow$ definability of bisimulation quantifiers.

A $\bar{p}$-model is a poset $(X, \leq)$, with a function $v \colon \bar{p} \to \mathrm{Up}(X, \leq)$.
By induction, any formula φ gets a semantics $[\![\varphi]\!]_X \in \mathrm{Up}(X, \leq)$.

If $E_p\varphi$ and $A_p\varphi$ are the uniform interpolants for φ, then

$$[\![E_p\varphi]\!]_X = \{x \in X \mid \exists X', x' \text{ with } (X', x') \sim_p (X, x) \text{ and } x' \in [\![\varphi]\!]_{X'}\},$$

$$[\![A_p\varphi]\!]_X = \{x \in X \mid \forall X', x' \text{ with } (X', x') \sim_p (X, x),\ x' \in [\![\varphi]\!]_{X'}\}.$$

Definable bisimulation quantifiers

First main idea in all semantic proofs (see also Visser, 1996):

uniform interpolation $\leftrightarrow$ definability of bisimulation quantifiers.

A $\bar{p}$-model is a poset $(X, \leq)$, with a function $v \colon \bar{p} \to \mathrm{Up}(X, \leq)$.
By induction, any formula φ gets a semantics $[\![\varphi]\!]_X \in \mathrm{Up}(X, \leq)$.

If $E_p\varphi$ and $A_p\varphi$ are the uniform interpolants for φ, then

$$[\![E_p\varphi]\!]_X = \{x \in X \mid \exists X', x' \text{ with } (X', x') \sim_p (X, x) \text{ and } x' \in [\![\varphi]\!]_{X'}\},$$

$$[\![A_p\varphi]\!]_X = \{x \in X \mid \forall X', x' \text{ with } (X', x') \sim_p (X, x),\ x' \in [\![\varphi]\!]_{X'}\}.$$

Here, $\sim_p$ is the relation of bisimilarity up to p.

Definable bisimulation quantifiers

First main idea in all semantic proofs (see also Visser, 1996):

uniform interpolation $\leftrightarrow$ definability of bisimulation quantifiers.

A $\bar{p}$-model is a poset $(X, \leq)$, with a function $v \colon \bar{p} \to \mathrm{Up}(X, \leq)$.
By induction, any formula φ gets a semantics $[\![\varphi]\!]_X \in \mathrm{Up}(X, \leq)$.

If $E_p\varphi$ and $A_p\varphi$ are the uniform interpolants for φ, then

$$[\![E_p\varphi]\!]_X = \{x \in X \mid \exists X', x' \text{ with } (X', x') \sim_p (X, x) \text{ and } x' \in [\![\varphi]\!]_{X'}\},$$

$$[\![A_p\varphi]\!]_X = \{x \in X \mid \forall X', x' \text{ with } (X', x') \sim_p (X, x),\ x' \in [\![\varphi]\!]_{X'}\}.$$

Here, $\sim_p$ is the relation of bisimilarity up to p.
Thus, it suffices to show that the sets on the right are definable.

Topological proof

To establish that the bisimulation quantifiers are definable, one can use a layered version of bisimulation. In our work with Reggio, we view this as a metric on the canonical model $X(p, \bar{q})$:

$$d(x, y) \stackrel{\text{def}}{=} 2^{-\min\{|\varphi|_{\rightarrow} \text{ : exactly one of } x \text{ and } y \text{ is in } [\![\varphi]\!]\}}.$$

Here, $|\varphi|_{\rightarrow}$ is the maximum depth of nestings of $\rightarrow$ in φ.

Topological proof

To establish that the bisimulation quantifiers are definable, one can use a layered version of bisimulation. In our work with Reggio, we view this as a metric on the canonical model $X(p, \bar{q})$:

$$d(x, y) \stackrel{\text{def}}{=} 2^{-\min\{|\varphi|_{\to} \,:\, \text{exactly one of } x \text{ and } y \text{ is in } [\![\varphi]\!]\}}.$$

Here, $|\varphi|_{\to}$ is the maximum depth of nestings of $\to$ in φ.

We then show that the projection $\pi_p \colon X(p, \bar{q}) \twoheadrightarrow X(\bar{q})$ is open:

Topological proof

To establish that the bisimulation quantifiers are definable, one can use a layered version of bisimulation. In our work with Reggio, we view this as a metric on the canonical model $X(p, \bar{q})$:

$$d(x, y) \stackrel{\text{def}}{=} 2^{-\min\{|\varphi|_{\to} \text{ : exactly one of } x \text{ and } y \text{ is in } [\![\varphi]\!]\}}.$$

Here, $|\varphi|_{\to}$ is the maximum depth of nestings of $\to$ in φ.
We then show that the projection $\pi_p \colon X(p, \bar{q}) \twoheadrightarrow X(\bar{q})$ is open:

Lemma

For every $n \in \mathbb{N}$, there exists $R(n) \gg n$ such that $B(\pi(x), 2^{-R(n)}) \subseteq \pi[B(x, 2^{-n})]$.

The number $R(n)$ gives a computable bound on the $\to$-depth of uniform interpolants of formulas of $\to$-depth n.

Outlook

- Uniform interpolation is a fertile ground for exploration: proof-theoretic, semantic, and computational aspects.

Outlook

- Uniform interpolation is a fertile ground for exploration: proof-theoretic, semantic, and computational aspects. There is still plenty of work to do:
- Better understanding of connection between proof theory vs. semantic proofs.

Outlook

- Uniform interpolation is a fertile ground for exploration: proof-theoretic, semantic, and computational aspects. There is still plenty of work to do:
- Better understanding of connection between proof theory vs. semantic proofs.
- Studying & improving complexity (theoretical & practical).

Outlook

- Uniform interpolation is a fertile ground for exploration: proof-theoretic, semantic, and computational aspects. There is still plenty of work to do:
- Better understanding of connection between proof theory vs. semantic proofs.
- Studying & improving complexity (theoretical & practical).
- Uniform interpolation for other logics; in particular, **iGL**.

Outlook

- Uniform interpolation is a fertile ground for exploration: proof-theoretic, semantic, and computational aspects. There is still plenty of work to do:
- Better understanding of connection between proof theory vs. semantic proofs.
- Studying & improving complexity (theoretical & practical).
- Uniform interpolation for other logics; in particular, **iGL**.
- For logics without (uniform) interpolation, an interesting computational problem: compute (uniform) interpolants when they exist, if not, provide a witness that they cannot exist.

Thank you!