Machines, Models, Monoids, and Modal logic

Sam van Gool

University of Amsterdam and City College of New York

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Outline

1. Part I: Formal Languages, Automata, and Algebra
2. Part II: Duality and Varieties of Monoids
Recap from Part I

- Formal $\Sigma$-languages are subsets of $\Sigma^*$, the set of finite words over a finite alphabet $\Sigma$. 
Recap from Part I

- Formal $\Sigma$-languages are subsets of $\Sigma^*$, the set of finite words over a finite alphabet $\Sigma$.
- Finite-state automata (deterministic or not) describe the regular languages.
- Monadic second order logic also describes exactly the regular languages.
- First order logic describes a (strictly) smaller class of languages.
- The regular languages form a Boolean algebra with quotient operators.
- Every regular language $L$ defines a finite closed Boolean subalgebra $B(L)$.
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- Monoids are also somehow important
Recap from Part I

- Formal $\Sigma$-languages are subsets of $\Sigma^*$, the set of finite words over a finite alphabet $\Sigma$.
- Finite-state automata (deterministic or not) describe the *regular* languages.
- Monadic second order logic also describes exactly the *regular* languages.
- First order logic describes a (strictly) smaller class of languages.
- The regular languages form a Boolean algebra with quotient operators.
- Every regular language $L$ defines a *finite* closed Boolean subalgebra $B(L)$.
- Monoids are also somehow important (*but why?*)
Monoids

Examples

- The set $\Sigma^*$, with multiplication $u \cdot v := uv$. 

Monoids

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- For any set $P$, the set of functions from $P$ to itself, $(P \rightarrow P)$, with multiplication $f \cdot g := f \circ g$. 
Monoids

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- The set $\Sigma^*$, with multiplication $u \cdot v := uv$.
- For any set $P$, the set of functions from $P$ to itself, $(P \to P)$, with multiplication $f \cdot g := f \circ g$.
- In particular, an NFA $A = (Q, \Sigma, \delta)$ gives, for every $a \in \Sigma$, a function $\diamond_a$ in $(\mathcal{P}(Q) \to \mathcal{P}(Q))$, defined by

$$\diamond_a(R) := \{ q \mid q \xrightarrow{a} q' \text{ for some } q' \in R \}.$$
Exercises

1. Show that $\Sigma^*$ is a monoid.
2. Show that $(P \rightarrow P)$ is a monoid.
3. Show that $\Sigma^*$ is the free monoid on $\Sigma$, i.e., that for any monoid $M$ and any function $f : \Sigma \rightarrow M$, there is a unique homomorphism $\bar{f} : \Sigma^* \rightarrow M$ extending $f$.
4. Applying (3) to the function $\Diamond : \Sigma \rightarrow (\mathcal{P}(Q) \rightarrow \mathcal{P}(Q))$, give an explicit description of the function $\bar{\Diamond} : \Sigma^* \rightarrow (\mathcal{P}(Q) \rightarrow \mathcal{P}(Q))$.
5. (*) Show that $\mathcal{A}$ with initial states $I$ and final states $F$ accepts a word $w \in \Sigma^*$ if, and only if, $I \cap \bar{\Diamond}_w(F) \neq \emptyset$. 
Proposition

A $\Sigma$-language $L$ is regular if, and only if, there exists a homomorphism $\eta: \Sigma^* \rightarrow M$, with $M$ a finite monoid, such that $L = \eta^{-1}(R)$ for some $R \subseteq M$. 
Regular languages and monoids

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Proof ingredients.

- The exercises on the previous slide show how to build a monoid homomorphism from an NFA.
- For the converse, notice that a homomorphism from $\Sigma^*$ to a monoid ‘is’ a (deterministic) automaton.
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Today, we will see how these characterizations are connected to each other through Stone duality.
Outline Part II

1 Finite Duality and Regular Languages
   - Boolean algebras
   - Finite Stone duality
   - Duality for regular languages

2 Full Duality and Varieties
   - First-order logic and aperiodic monoids
   - Full Stone duality
“In January last year I gave a course at the Indian Winter School in Logic and went on an excursion to Varanasi and Sarnath, the birthplace of Buddhism. Upon entering the amazing Archaeological Museum at Sarnath, our guide opened with: ‘Dualities underlies the world.’ This is the kind of sweeping statement that every mathematician, at least secretly, would like to believe about their particular focus...”

M. Gehrke. *Duality*. Oratie (inaugural lecture) at Radboud University Nijmegen, 2009. URL: http://repository.ubn.ru.nl/bitstream/handle/2066/83300/83300.pdf
Stone duality

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- The **dual** of a collection of formulas (syntax) is a space of possible worlds/states (semantics) interpreting the formulas, and vice versa.

A key idea, and the meaning of the term ‘duality’ (= dual categorical equivalence), is that the direction of morphisms is reversed.

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Formulating duality theory precisely requires some algebra, and, for the non-finite case, topology.

We will focus on the applications to regular languages.
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Finite Duality and Regular Languages
- Boolean algebras
- Finite Stone duality
- Duality for regular languages
Boolean algebras

- An (abstract) *Boolean algebra* is a tuple \((B, \lor, \neg, \bot)\), where
  - \(B\) is a set,
  - \(\lor\) is a binary operation,
  - \(\neg\) is a unary operation,
  - \(\bot\) is an element of \(B\),
  - for any classical tautology \(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x})\) and \(\bar{b}\) in \(B\), \(\varphi(\bar{b}) = \psi(\bar{b})\) in \(B\).
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- Boolean algebras are partially ordered: \(a \leq b\) iff \(a \lor b = b\).
Boolean algebras: examples

Examples

- For any set $X$, $(\mathcal{P}(X), \cup, (\cdot)^c, \emptyset)$ is a Boolean algebra.
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- The *Lindenbaum algebra* of classical propositional logic on a set of variables $V$ is the *free* Boolean algebra on $V$. 
Examples

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- The *Lindenbaum algebra* of classical propositional logic on a set of variables $V$ is the *free* Boolean algebra on $V$.
- For any topological space $X$, the *clopen* (= closed and open) subsets are a Boolean subalgebra of $\mathcal{P}(X)$. 
Finite Stone duality: algebras

Proposition

Every finite Boolean algebra $B$ is isomorphic to a Boolean algebra of the form $\mathcal{P}(X)$, for some set $X$.
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Take $X = \text{At}(B)$, the set of atoms of $B$.
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**Proof.**

Take $X = \text{At}(B)$, the set of atoms of $B$. Identify $b \in B$ with the set, $\hat{b}$, of atoms below it.

Example

If $V = \{p_1, \ldots, p_n\}$, then the Lindenbaum algebra of classical propositional logic on $V$ is isomorphic to $\mathcal{P}(X)$, where $X = \{0, 1\}$.

In words: a formula of CPL can be identified with the set of valuations in which it is true. When $V$ is infinite, the situation is more subtle!
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Finite Stone duality: homomorphisms

**Proposition**

*Every homomorphism between finite Boolean algebras* $\mathcal{P}(Y) \to \mathcal{P}(X)$ *is of the form* $f^{-1}$ *for some function* $f : X \to Y$.
Finite Stone duality: homomorphisms

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Every homomorphism between finite Boolean algebras $\mathcal{P}(Y) \to \mathcal{P}(X)$ is of the form $f^{-1}$ for some function $f : X \to Y$.

- In particular, any finite subalgebra of $\mathcal{P}(X)$ has the form $q^{-1} : \mathcal{P}(Y) \hookrightarrow \mathcal{P}(X)$, where $q : X \twoheadrightarrow Y$ is a quotient of $X$.
- In other words, any finite subalgebra of $\mathcal{P}(X)$ is the collection of finite unions of equivalence classes of an equivalence relation on $X$. 
Subalgebras and equivalence relations

Example

- The closed subalgebra generated by the $\Sigma$-language $L = \text{EVENLENGTH}$ is

$$B(L) = \{\emptyset, L, L^c, \Sigma^*\} \hookrightarrow \text{Reg}(\Sigma^*).$$
Subalgebras and equivalence relations

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- The closed subalgebra generated by the $\Sigma$-language $L = \text{EVENLENGTH}$ is

  $$B(L) = \{\emptyset, L, L^c, \Sigma^*\} \hookrightarrow \text{Reg}(\Sigma^*).$$

- The dual of this subalgebra is a quotient $q: \Sigma^* \rightarrow \text{At } B(L)$.

- This quotient is given by the equivalence relation $w_1 \equiv_L w_2$ if, and only if, the length of $w_1$ and $w_2$ have the same parity.
Let $L$ be a regular $\Sigma$-language.

\[ B(L) \] is the set of unions of equivalence classes under an equivalence relation $\equiv_L$ on $\Sigma^\ast$, which can be defined by

\[ w_1 \equiv_L w_2 \iff \text{for all } u, v \in \Sigma^\ast, uw_1v \in L \text{ iff } uw_2v \in L. \]

A language $L \subseteq \Sigma^\ast$ is regular if, and only if, $\equiv_L$ has finite index.
Finite Stone duality: regular languages

- Let $L$ be a regular $\Sigma$-language.
- Let $B(L)$ be the finite closed subalgebra of $\text{Reg}(\Sigma^*)$ generated by $L$. 
Finite Stone duality: regular languages

- Let $L$ be a regular $\Sigma$-language.
- Let $B(L)$ be the finite closed subalgebra of $\text{Reg}(\Sigma^*)$ generated by $L$.
- Then $B(L)$ is the set of unions of equivalence classes under an equivalence relation $\equiv_L$ on $\Sigma^*$.
Finite Stone duality: regular languages

Let $L$ be a regular $\Sigma$-language.

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Then $B(L)$ is the set of unions of equivalence classes under an equivalence relation $\equiv_L$ on $\Sigma^*$, which can be defined by

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Duality and regular languages

- $B(L)$ is a *closed* subalgebra of $\text{Reg}(\Sigma^*)$. 
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- $B(L)$ is a *closed* subalgebra of $\text{Reg}(\Sigma^*)$.
- It follows that the dual $M(L) = \Sigma^*/\equiv_L$ of $B(L)$ is a *monoid*. 

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Duality and regular languages

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- The monoid $M(L)$ is the \textit{syntactic monoid} of $L$.
- The \textbf{homomorphism} $q: \Sigma^* \rightarrow M(L)$ \textit{recognizes} $L$:
  $L = q^{-1}(R)$ where $R = q(L)$.
**Duality and regular languages**

- $B(L)$ is a *closed* subalgebra of $\text{Reg}(\Sigma^*)$.
- It follows that the dual $M(L) = \Sigma^*/\equiv_L$ of $B(L)$ is a *monoid*.
- The monoid $M(L)$ is the *syntactic monoid* of $L$.
- The homomorphism $q: \Sigma^* \rightarrow M(L)$ *recognizes* $L$:
  \[ L = q^{-1}(R) \text{ where } R = q(L). \]
- Moreover, $M(L)$ is the *minimum* such monoid quotient of $\Sigma^*$:
  if $q': \Sigma^* \rightarrow M'$ recognizes $L$, then there exists $f: M' \rightarrow M(L)$ such that $fq' = q$. 

\[ v. \text{ Gool (UvA & CCNY)} \]
Syntactic monoid: Example

Example

Let $\Sigma = \{0, 1\}$ and $L = \text{EVENLENGTH}$. 
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Let $\Sigma = \{0, 1\}$ and $L = \text{EVENLENGTH}$. For $w_1, w_2 \in \Sigma^*$, $w_1 \equiv_L w_2$ iff the length of $w_1$ and of $w_2$ have the same parity. Therefore, $M(L) \cong \mathbb{Z}_2$, the two-element group. The quotient $q: \Sigma^* \rightarrow M(L)$ is defined by $q(w) := \text{parity of the length of } w$. Notice that $q(w_1w_2) = q(w_1) \oplus q(w_2)$, i.e., $q$ is a homomorphism.
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Exercises

1. Find the syntactic monoid quotient $\Sigma^* \rightarrow M(L)$ when $L = \text{EVENONES}$.

2. Find the syntactic monoid quotient $\Sigma^* \rightarrow M(L)$ when $L = \text{BUY}$.

3. (*) Find the syntactic monoid quotient $\Sigma^* \rightarrow M(L)$ when $L = \text{PW}$.

4. Conclude from the solutions to (1) – (3) what the closed subalgebras, $B(L)$, generated by $L$ are.

5. Use $\equiv L$ to show that $L$ is not regular when $L = \text{N0N1}$.
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4. Conclude from the solutions to (1) – (3) what the closed subalgebras, $B(L)$, generated by $L$ are. 
5. Use $\equiv_L$ to show that $L$ is not regular when $L = \text{NON1}$. 
2 Full Duality and Varieties

- First-order logic and aperiodic monoids
- Full Stone duality
FO and aperiodics

- In Part I, we asked: what is the subalgebra $\text{FO}(\Sigma^*)$ of $\text{Reg}(\Sigma^*)$?

A monoid $M$ is aperiodic if it contains no non-trivial subgroups. For finite monoids, it is equivalent to say:

- the equation $x^n = x^{n+1}$ holds in $M$ for some $n$.
- $x^\omega = x^\omega x^\omega$, where $x^\omega$ is the idempotent power of $x$.

Theorem (Schützenberger, 1960s)

A language $L$ is first-order definable if, and only if, the syntactic monoid $M(L)$ is finite and aperiodic.

An algorithm for deciding if a regular language is FO-definable.
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We now know that any regular language $L$ has a finite syntactic monoid $M(L)$.
FO and aperiodics

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- This contains (in fact, is) a group.
- By Schützenberger’s theorem, EVENLENGTH is not first order definable.
Exercise

- Using the results from the previous exercise, determine which of the syntactic monoids for \textsc{EVENONES}, \textsc{BUY}, and \textsc{PW} are aperiodic.
- Conclude which of these languages are first order definable.
Varieties of monoids and languages

- A class of finite monoids \( V \) is a (pseudo)\( \textit{variety} \) if it is closed under homomorphic images (H), submonoids (S) and finite products (\( P^{\text{fin}} \)).
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**Theorem (Eilenberg)**

The map $\mathbf{V} \leftrightarrow \mathcal{V}$ is an order-bijection between varieties of finite monoids and varieties of regular languages.
Equations?

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- To explain what these are, and why we need them: full Stone duality.
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Every Boolean algebra $B$ can be embedded into a Boolean algebra of the form $\mathcal{P}(X)$.
Stone duality: general case

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Every Boolean algebra $B$ can be embedded into a Boolean algebra of the form $\mathcal{P}(X)$, and there is a unique such embedding for which the topology generated by the sets in the image of $B$ is compact and Hausdorff (and zero-dimensional).
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- A *Boolean space* is a compact Hausdorff zero-dimensional space.
- Equivalently, a *Boolean space* is a profinite object in the category of topological spaces.
Stone duality: example

Example
The dual space of the Lindenbaum algebra of CPL on a countable set $V = \{p_1, p_2, p_3, \ldots \}$ is the Cantor space $\{0, 1\}^V$. 
Exercises

1. What is the dual space of the Boolean algebra of finite subsets of the natural numbers and their complements?

2. Use what you know about classical propositional logic to prove that the Lindenbaum algebra of CPL on a countable set \( V = \{p_1, p_2, p_3, \ldots \} \) can be embedded into \( \mathcal{P}(\{0, 1\}^V) \).

3. (*) Show that the topology generated by the image of the embedding in (2) is compact and Hausdorff.

4. (*) Show that the topology generated by the image of the embedding in (2) coincides with the topology of the Cantor space.
Duality: categorical level

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- Our recent work on applications of model theory: