Note. Below you find one possible solution to each problem; other correct solutions are often possible.

1. Let \( n \) be an integer.
   
   (a) Prove: if \( 5n + 3 \) is odd then \( n \) is even.
   
   We give a proof by contrapositive. Suppose that \( n \) is odd. Then \( n = 2x + 1 \) for some integer \( x \). Therefore,
   
   \[
   5n + 3 = 5(2x + 1) + 3 = 10x + 8 = 2(5x + 4),
   \]
   
   which is even, because \( 5x + 4 \) is an integer.

   (b) Prove, using item (a): if \( 5n + 3 \) is odd then \( 3n - 4 \) is even.

   Assume that \( 5n + 3 \) is odd. By item (a), \( n \) is even. Thus, \( n = 2y \) for some integer \( y \).

   Therefore,
   
   \[
   3n - 4 = 6y - 4 = 2(3y - 2),
   \]
   
   which is even, because \( 3y - 2 \) is an integer.

   (c) Give the missing word in the following sentence: “When we proved the result in (b), we used the result in (a) as a __________.”

   Lemma.

2. State the following two statements in words:
   
   (a) \( \exists x \in \mathbb{R}, x^2 = -1. \)

   There exists a real number \( x \) such that \( x^2 \) is equal to \(-1\).

   (b) \( 3 \mid a \iff a \equiv 2 \pmod{4}. \)

   3 divides \( a \) if and only if \( a \) is congruent to 2 modulo 4.

   State the negation of the following two statements in words:

   (c) John is driving only if John is wearing a seatbelt.

   John is driving and John is not wearing a seatbelt. or:

   It is not the case that John is driving only if John is wearing a seatbelt.

   (d) For every natural number \( n \), if \( n \) is odd, then \( n^2 \) is odd or \( n - 3 \) is even.

   There exists a natural number \( n \) such that \( n \) is odd, \( n^2 \) is even, and \( n - 3 \) is odd. or:

   It is not the case that for every natural number \( n \), if \( n \) is odd, then \( n^2 \) is odd or \( n - 3 \) is even.

3. Let \( A, B, \) and \( C \) be subsets of a universal set \( U \).

   (a) For \( U = \mathbb{N} \), give an example of three sets \( A, B \) and \( C \) that are pairwise disjoint.

   We need an example of sets of natural numbers \( A, B, C \) such that the three sets \( A \cap B, A \cap C \) and \( B \cap C \) are empty. For example, \( A = \{1\}, B = \{2, 3, 4\} \) and \( C = \{5, 6, 7\} \).

   (b) Prove that \( A - (B \cap C) = (A \cap \overline{B}) \cup (A \cap \overline{C}) \). You may use any proof method you wish. If you use any laws, state their names.

   \[
   A - (B \cap C) = A \cap B \cap \overline{C} \quad \text{ (since } A - D = A \cap \overline{D})
   \]

   \[
   = A \cap (B \cup \overline{C}) \quad \text{ (by de Morgan’s law)}
   \]

   \[
   = (A \cap B) \cup (A \cap \overline{C}) \quad \text{ (by the distributive law)}.
   \]
4. Let \( x \) and \( y \) be real numbers.
   (a) Prove that, if \( x^3 + y - xy \geq 0 \), then \( x \geq 0 \) or \( y \geq 0 \).
   Assume that \( x < 0 \) and \( y < 0 \). Then \( x^3 < 0 \) and \( xy > 0 \), so \(-xy < 0\). Therefore, \( x^3 + y - xy < 0 \).
   (b) Give the name of the proof method you used in item (a).
   Proof by contrapositive.

5. For each \( k \in \{0, 1, 2, 3\} \), let \( A_k \) be the set \( \{x \in \mathbb{Z} \mid x \equiv k \mod 4\} \).
   (a) Is \( \{A_0, A_1, A_2, A_3\} \) a partition of \( \mathbb{Z} \)? You should state the definition of ‘partition’ to explain your answer.
   Yes. It satisfies the three conditions for being a partition, namely, the sets \( A_k \) are pairwise disjoint (\( A_k \cap A_\ell = \emptyset \) when \( k \neq \ell \)), their union is \( \mathbb{Z} \) (\( \bigcup_{k=0}^{3} A_k = \mathbb{Z} \)), and none of the sets is empty (\( A_k \neq \emptyset \) for each \( k \)).
   (b) Prove that, for every integer \( y \): \( y^2 \in A_1 \) if and only if \( y \in A_1 \) or \( y \in A_3 \). You may use the following fact without proving it: for any integers \( x \) and \( y \), if \( y \equiv x \mod 4 \), then \( y^2 \equiv x^2 \mod 4 \).
   Let \( y \) be an integer. First assume that \( y \in A_1 \) or \( y \in A_3 \).
   Case 1. \( y \in A_1 \). By definition, \( y \equiv 1 \mod 4 \). By the fact, \( y^2 \equiv 1 \mod 4 \), so \( y^2 \in A_1 \).
   Case 2. \( y \in A_3 \). By definition, \( y \equiv 3 \mod 4 \). By the fact, \( y^2 \equiv 9 \mod 4 \). Since \( 9 \equiv 1 \mod 4 \), \( y^2 \in A_1 \).
   Conversely, assume that \( y \notin A_1 \) and \( y \notin A_3 \). There are two cases: \( y \in A_0 \) or \( y \in A_2 \).
   Case 1. \( y \in A_0 \). Then \( y \equiv 0 \mod 4 \), so \( y^2 \equiv 0 \mod 4 \) by the fact. So \( y^2 \in A_0 \). Hence, \( y^2 \notin A_1 \).
   Case 2. \( y \in A_2 \). Then \( y \equiv 2 \mod 4 \), so \( y^2 \equiv 4 \mod 4 \) by the fact. Since \( 4 \equiv 0 \mod 4 \), it follows that \( y^2 \in A_0 \). Hence, \( y^2 \notin A_1 \).

6. Let \( P \), \( Q \) and \( R \) be statements.
   (a) Prove, using a truth table, that \( P \Rightarrow (Q \Rightarrow P) \) is a tautology.
   \[
   \begin{array}{c|c|c|c}
   P & Q & Q \Rightarrow P & P \Rightarrow (Q \Rightarrow P) \\
   \hline
   T & T & T & T \\
   T & F & T & T \\
   F & T & F & T \\
   F & F & T & T \\
   \end{array}
   \]
   Since the truth values in the column \( P \Rightarrow (Q \Rightarrow P) \) are all \( T \), it is a tautology.
   (b) Prove, without using a truth table, that \( P \Rightarrow \neg(Q \land R) \equiv \neg P \lor \neg Q \lor \neg R \). (Hint: use the fact that, for any statements \( P \) and \( S \), \( P \Rightarrow S \equiv \neg P \lor S \), and use de Morgan’s laws. Clearly state when you use these facts.)
   \[
   P \Rightarrow \neg(Q \land R) \equiv \neg P \lor \neg(Q \land R) \quad \text{(using } P \Rightarrow S \equiv \neg P \lor S) \\
   \equiv \neg P \lor \neg Q \lor \neg R. \quad \text{(using de Morgan’s law)}
   \]