1. Read the following start of a proof; the questions follow the text.

**Theorem.** Let $A$ and $B$ be finite sets such that $|A| = n = |B|$. Define the set 
$\text{Bi}(A, B) := \{f : A \to B : f \text{ is bijective}\}$. Then $|\text{Bi}(A, B)| = n!$.

**Proof.** We prove, by induction on $n$, that $\forall n, P(n)$, where $P(n)$ is the statement

$$P(n) : \text{for any finite sets } A, B \text{ such that } |A| = n = |B|, |\text{Bi}(A, B)| = n!.$$ 

**Base case.** If $n = 1$, write $A = \{a\}$ and $B = \{b\}$. Then there is one bijection from $A$ to $B$, namely $\{(a, b)\}$. So $|\text{Bi}(A, B)| = 1 = 1!$.

**Induction step.** Suppose $P(k)$ holds. We prove $P(k + 1)$. Let $A$ and $B$ be finite sets such that $|A| = k + 1 = |B|$. Pick an element $a$ of $A$. For every $b$ in $B$, define the set 
$\text{Bi}(A, B)_{a \mapsto b} := \{f \in \text{Bi}(A, B) : f(a) = b\}$.

(a) For example, let $A = \{1, 2, 3\}$ and $B = \{x, y, z\}$. List the two bijections that are in the set $\text{Bi}(A, B)_{1 \mapsto x}$?

(b) In the example from (a), is the collection $\{\text{Bi}(A, B)_{1 \mapsto x}, \text{Bi}(A, B)_{1 \mapsto y}, \text{Bi}(A, B)_{1 \mapsto z}\}$ a partition of $\text{Bi}(A, B)$?

We now return to the general case and finish the proof.

(c) Prove that the collection $\mathcal{S} := \{\text{Bi}(A, B)_{a \mapsto b} : b \in B\}$ is a partition of $\text{Bi}(A, B)$ into $k + 1$ classes.

(d) Explain why, for every $b \in B$, $|\text{Bi}(A, B)_{a \mapsto b}| = |\text{Bi}(A \setminus \{a\}, B \setminus \{b\})|$.

(e) Using items (c) and (d) and the induction hypothesis, $P(k)$, prove that $|\text{Bi}(A, B)| = (k + 1) \cdot k!$.

which is equal to $(k + 1)!$.

2. Let $f$ be the relation from $\{1, 2, 3, 4\}$ to $\mathbb{Q}$ defined by

$$f := \{(1, \frac{1}{2}), (2, \frac{2}{5}), (3, \frac{5}{3}), (4, \frac{2}{7})\}.$$

(a) Prove that $f$ is a function.
(b) Prove or disprove: $f$ is one-to-one.

(c) Prove or disprove: $f$ is onto.

(d) Let $A := \{x \in \mathbb{Q} : 0 \leq x \leq \frac{1}{2}\}$. List the elements of the inverse image set, $f^{-1}(A)$.

3. Prove or disprove, for any functions $f : A \to B$ and $g : B \to C$,

(a) If $g \circ f$ is injective, then $f$ is injective.

(b) If $g \circ f$ is surjective, then $f$ is surjective.

4. Let $f : [0, 2] \to [3, 7]$ be the function defined by $f(x) := x^2 + 3$.

(a) Prove that $f$ is injective.

(b) Prove that $f$ is surjective.

(c) Define the inverse function $g : [3, 7] \to [0, 2]$ of $f$.

5. For every natural number $n$, let $S_n$ be the set of permutations on $\{1, \ldots, n\}$, let $T_n$ be the set of subsets of $\{1, \ldots, n\}$ which do not have cardinality $n - 1$, and, for every $\alpha \in S_n$, let $f_n(\alpha)$ be the set of fixpoints of $\alpha$. In other words,

$$S_n := \{\alpha \in \{1, \ldots, n\}^{\{1, \ldots, n\}} : \alpha \text{ is bijective}\},$$

$$T_n := \{F \in \mathcal{P}(\{1, \ldots, n\}) : |F| \neq n - 1\},$$

$$f_n(\alpha) := \{k \in \{1, \ldots, n\} : \alpha(k) = k\}.$$

(a) Prove that, for every $\alpha \in S_n$, $f_n(\alpha) \in T_n$. (Hint. There are two cases: either $\alpha$ is the identity function, or $\alpha$ is different from the identity function.)

(b) Prove that $f_n : S_n \to T_n$ is onto for every natural number $n$.

(Hint. You may use as a Lemma that for any finite set $S$, if $|S| \geq 2$, then there exists a permutation $\alpha$ on $S$ such that, for any $s \in S$, $\alpha(s) \neq s$.)

(c) Prove that $|T_n| = 2^n - n$ for every natural number $n$.

(Hint. You may use as a Lemma that $|\mathcal{P}(\{1, \ldots, n\})| = 2^n$.)

(d) Use items (b), (c), and the fact that $|S_n| = n!$ to prove that $n! \geq 2^n - n$ for every natural number $n$. 